Highly accurate solutions of Motz’s and the cracked beam problems

T.T. Lu, H.Y. Hu, Z.C. Li*

Department of Applied Mathematics and Department of Computer Science and Engineering, National Sun Yat-Sen University, Kaohsiung 80424, Taiwan, ROC

Received 17 June 2003; revised 30 January 2004; accepted 21 March 2004

Available online 13 July 2004

Abstract

For Motz’s problem and the cracked beam problem, the collocation Trefftz method is used to seek their approximate solutions \( u_N = \sum_{i=0}^{\infty} D_i r^{i+1/2} \cos(i + (1/2)) \theta \), where \( D_i \) are the expansion coefficients. The high-order Gaussian rules and the central rule are used in the algorithms, to link the collocation method and the least squares method, and to provide exponential convergence rates of the obtained solutions. Compared with the solutions in the previous literature, our Motz’s solutions are more accurate and the leading coefficient algorithms, to link the collocation method and the least squares method, and to provide exponential convergence rates of the obtained solutions. The exponential convergence rates \( O(e^{-cN}) \) can be obtained for Eq. (1.5) with some positive constant \( c \). When function (1.5) is chosen, Eq. (1.1), \( u|_{x=0,y=0} = 0 \) and \( u|_{x=1} = 500 \) are satisfied automatically. Then the coefficients \( D_i \) are sought by the collocation equations of the rest boundary conditions in Eqs. (1.2) and (1.3). This is called the boundary approximation method (BAM) in Refs. [14,16] or the collocation Trefftz method in this paper. Under the computation in double precision and \( N = 34 \), the maximal absolute error at \( x = 1 \) (e.g. on \( \overline{AB} \)) of

1. Introduction

Motz’s problem was first discussed by Motz [18] in 1947 for the relaxation method. Since then, many researchers have selected Motz’s problem as a prototype of singularity problems for verifying efficiency of numerical methods [14]. Motz’s problem solves the Laplace equation on the rectangle \( S = \{(x,y)| -1 < x < 1, 0 < y < 1\} \)

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \text{ in } S, \tag{1.1}
\]

with the mixed Neumann–Dirichlet boundary conditions, Fig. 1

\[
u|_{x=0,y=0} = 0, \quad u|_{x=1} = 500, \tag{1.2}
\]

\[
\frac{\partial u}{\partial y}|_{y=1} = \frac{\partial u}{\partial y}|_{y=0} = \frac{\partial u}{\partial x}|_{x=-1} = 0. \tag{1.3}
\]

Note that there exists a singularity at the origin (0,0) due to the intersection of the Neumann–Dirichlet boundary conditions. In fact, the singular solutions of Eqs. (1.1)–(1.3) are found as

\[
u_{c}(r, \theta) = \sum_{i=0}^{\infty} d_i r^{i+1/2} \cos(i + (1/2)) \theta, \tag{1.4}
\]

where \( d_i \) is the true expansion coefficient, and \( (r, \theta) \) are the polar coordinates with the origin at (0,0) (Fig. 1). Since its convergence radius, \( R = 2 \), is analyzed in Ref. [20], the series expansions (1.4) are well suited to the entire solution domain \( S \). Hence, the admissible functions with finite terms

\[
u_{N}(r, \theta) = \sum_{i=0}^{N} D_i r^{i+1/2} \cos(i + (1/2)) \theta, \tag{1.5}
\]

where \( D_i \) are the unknown coefficients, are most efficient as numerical Motz’s solutions. The exponential convergence rates \( O(e^{-cN}) \) can be obtained for Eq. (1.5) with some positive constant \( c \). When function (1.5) is chosen, Eq. (1.1), \( u|_{x=0,y=0} = 0 \) and \( \frac{\partial u}{\partial y}|_{y=0} = 0 \) are satisfied automatically. Then the coefficients \( D_i \) are sought by the collocation equations of the rest boundary conditions in Eqs. (1.2) and (1.3). This is called the boundary approximation method (BAM) in Refs. [14,16] or the collocation Trefftz method in this paper. Under the computation in double precision and \( N = 34 \), the maximal absolute error at \( x = 1 \) (e.g. on \( \overline{AB} \)) of

* Corresponding author.
E-mail address: zeli@math.nsysu.edu.tw (Z.C. Li).
the Motz’s solution in Ref. [16] reaches up to $5.47 \times 10^{-9}$. Also the leading coefficient $D_0$ in Ref. [16] has 12 significant digits. The solutions in Ref. [16] have been recognized to be the very accurate solutions for Motz’s problem [5,6,17]. In this paper, to pursue the better leading coefficient $D_0$, we choose the Gaussian rules of high orders. Surprisingly, the obtained $D_0$ may have 17 significant digits. The solutions in Ref. [16] have been recognized to be the very accurate solutions for Motz’s problem over Motz’s problem is that half of the expansion coefficients are zero.

The same approaches are applied to the cracked beam problem, which is another frequently used model for testing new numerical methods [4–6,19,21]. Its highly accurate solutions are also provided with the leading coefficient $D_0$ having 17 significant digits. The advantage of the cracked beam problem over Motz’s problem is that half of the expansion coefficients are zero.

This paper is organized as follows. In Section 2, basic algorithms of the collocation Trefftz method are provided for Motz’s problem, and the highly accurate solutions are obtained in double precision. In Section 3, a new analysis is made for the quadrature involved. In Section 4, the cracked beam problem is discussed, and its highly accurate solutions and the leading coefficient $D_0$ with 17 significant digits are also reported. In Section 5, some discussions and comparisons are made, and in Section 6, concluding remarks are addressed.

2. Basic algorithms of collocation Trefftz method

Since the expansion (1.5) satisfies the Laplace equation and boundary conditions at $y = 0$, the coefficients $D_i$ should be chosen to satisfy the rest of the boundary conditions

$$u_{|y=1} = u_N = 500,$$  

$$\frac{\partial u}{\partial y}_{|y=1} = \frac{\partial u}{\partial y}_{|y=-1}, \quad \frac{\partial u}{\partial x}_{|y=1} = \frac{\partial u}{\partial x}_{|y=-1} = 0.$$  

as best as possible, where $u_x = \partial u/\partial v$ is the outward normal derivative to $\partial S$, and $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ are shown in Fig. 1. Hence, the least squares method (LSM) may be designed as follows. Denote

$$[u, v] = \int_{\overline{AB}} u v \, dl + w^2 \int_{\overline{BC} \cup \overline{CD}} u v_x \, dl,$$  

where $w$ is a positive weight constant, and a good choice of the weight

$$w = \frac{1}{N+1},$$  

can be found in Ref. [16]. Denote by $V_N$ the collection of finite dimensional function (1.5). Then, we may seek $u_N \in V_N$ such that

$$[u_N, v] = (f, v), \quad \forall v \in V_N,$$  

where

$$(f, v) = 500 \int_{\overline{AB}} v \, dl.$$  

Denote the energy

$$I(v) = \int_{\overline{AB}} (v - 500)^2 \, dl + w^2 \int_{\overline{BC} \cup \overline{CD}} v_x^2 \, dl.$$  

The solution of Eq. (2.5) can also be expressed by: to seek $u_N \in V_N$ such that

$$I(u_N) = \min_{v \in V_N} I(v).$$  

Both Eqs. (2.5) and (2.8) lead to the same linear algebraic system

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$  

where $\mathbf{x} \in \mathbb{R}^{N+1}$ is the unknown vector consisting of coefficients $D_i$, $i = 0, \ldots, N$, and $\mathbf{b} \in \mathbb{R}^{N+1}$ is the known vector resulting from the non-homogeneous Dirichlet condition (2.1), and the associate matrix, $\mathbf{A} \in \mathbb{R}^{(N+1)(N+1)}$, is symmetric positive definite, but not sparse. By the Gaussian elimination without pivoting in Ref. [7], the coefficients $D_i$ (i.e. $\mathbf{x}$) can be obtained. Once the coefficients $D_i$ are known, the errors on $\overline{AB} \cup \overline{BC} \cup \overline{CD}$

$$||u - u_N||_B = \left[ \int_{\overline{AB}} (500 - u_N)^2 \, dl + w^2 \int_{\overline{BC} \cup \overline{CD}} (u_N^2) \, dl \right]^{1/2}$$  

are computable, where the notation is

$$||v||_B = \sqrt{\langle v, v \rangle}.$$  

Suppose that certain rules of integration are adopted to the integrals in Eq. (2.7). Let $\overline{AB}$ be divided into small segments
Then the integral is evaluated by some rules
\[ \int_{\overline{AB}} v^2 \, dl = \sum_i \int_{v_i}^{v_{i+1}} v^2 \, dl. \] (2.12)

For example, the central and trapezoidal rules are given by
\[ \int_{\overline{AB}} v^2 \, dl = \frac{1}{2} (v_i^2 + v_{i+1}^2) h_i, \] (2.13)
and
\[ \int_{\overline{AB}} v^2 \, dl = \frac{1}{2} (v_i^2 + v_{i+1}^2) h_i, \] (2.14)
respectively, where \( h_i = \overline{Z_iZ_{i+1}}, \ v_i = v(Z_i), \ v_{i+1} = v(Z_{i+1}) \) and \( Z_{i+1} = Z_i + Z_{i+1}/2. \) Other kinds of Newton–Cotes and Gaussian rules can also be employed and will be discussed later. Hence, for the numerical quadrature, we may seek \( \mathbf{u}_N \in V_N \) such that
\[ \mathbf{I}(\mathbf{u}_N) = \min_{\mathbf{v} \in V_N} \mathbf{I}(\mathbf{v}), \] (2.15)
where
\[ \mathbf{I}(\mathbf{v}) = \sum_i \int_{\overline{AB}} (v - 500)^2 \, dl + \omega^2 \int_{BC \cup CD} v^2 \, dl. \] (2.16)

The minimization of \( \mathbf{I}(\mathbf{v}) \) also leads to a linear system like Eq. (2.9). This is a direct implementation to the LSM involving numerical integration, called the normal method.

Now, we turn to the collocation Trefftz method, which can be regarded as a certain kind of the LSM involving specific quadratures. For simplicity in exposition, let us first consider the central rule (2.13). Divide the boundary \( \overline{AB}, \overline{BC} \) and \( \overline{CD} \) into uniform sub-intervals (Fig. 1). Then
\[ h = \frac{\overline{AB}}{M} = \frac{\overline{CD}}{M} = \frac{\overline{AB}}{2M}. \] (2.17)
Eqs. (2.1) and (2.2) can be transformed to the boundary collocation equations
\[ u_N(P_i) = 500, \quad i = 1, 2, \ldots, M, \] (2.18)
\[ \frac{\partial u_N}{\partial x}(P_i^*) = -\frac{\partial u_N}{\partial v}(P_i^*) = 0, \quad i = 1, 2, \ldots, M, \] (2.19)
\[ \frac{\partial u_N}{\partial y}(Q_i^*) = \frac{\partial u_N}{\partial v}(Q_i^*) = 0, \quad i = 1, 2, \ldots, M. \] (2.20)

\[ P_i = (r_i, \theta_i), \quad r_i = \sqrt{1 + y_i}, \quad \theta_i = \cos^{-1} \left( \frac{1}{\sqrt{1 + y_i}} \right), \]
\[ P_i^* = (r_i, \theta_i^*), \quad \theta_i^* = \pi - \theta_i, \]
where \( 0 < \theta_i < \pi/2. \) Besides
\[ Q_i^* = (r_i, \theta_i^*), \quad r_i = \sqrt{1 + x_i^2}, \quad \theta_i^* = \sin^{-1} \left( \frac{1}{\sqrt{1 + x_i^2}} \right), \]
where \( 0 < \theta_i^* < \pi/2 \) and \( \theta_i^* = \pi - \theta_i^* \)

In Eqs. (2.18)–(2.20), there are \( m = 4M \) equations, but \( N + 1 \) unknown coefficients. Usually, select \( m > N + 1. \) We invoke the standard least squares method in Ref. [7] to solve the overdetermined system of Eqs. (2.18)–(2.20). Denote Eqs. (2.18)–(2.20) by
\[ \mathbf{F}_i \bar{x} = \bar{b}_i, \quad i = 1, 2, 3, \] (2.21)
respectively, where \( \mathbf{F}_i \) and \( \bar{b}_i \) are the known matrices and vectors, respectively. Since Eq. (2.21) results from different boundary conditions, different weights should also be assigned. When the weights \( \sqrt{h} \) and \( w\sqrt{h} \) are applied to the first and the other two equations in Eq. (2.21), the global target function becomes
\[ T(\bar{x}) = h \| \mathbf{F}_1 \bar{x} - \bar{b}_1 \|^2 + w^2 \sum_{i=2}^3 \| \mathbf{F}_i \bar{x} - \bar{b}_i \|^2, \] (2.22)
where \( \| \cdot \| \) is the Euclidean norm, and \( w \) is a suitable weight constant (Eq. (2.4)). We can easily verify the following lemma by direct manipulation.

**Lemma 2.1.** Let the central rule (2.13) be used in Eqs. (2.16) and (2.21) be the collocation equations (2.18)–(2.20). Then we have
\[ \mathbf{I}(\mathbf{u}_N) = T(\bar{x}), \] (2.23)
where \( \mathbf{I}(\mathbf{u}_N) \) and \( T(\bar{x}) \) are defined in Eqs. (2.16) and (2.22), respectively.

Note that the admissible functions and their derivatives are given by
\[ u_N = u_N(r, \theta) = \sum_{l=0}^N D_l r^{l+(1/2)} \cos \left( l + \frac{1}{2} \right) \theta, \]
\[ \frac{\partial u_N}{\partial x} = \sum_{l=0}^N D_l \left( l + \frac{1}{2} \right) r^{l-(1/2)} \cos \left( l - \frac{1}{2} \right) \theta, \]
\[ \frac{\partial u_N}{\partial y} = \sum_{l=0}^N D_l \left( l + \frac{1}{2} \right) r^{l-(1/2)} \sin \left( l - \frac{1}{2} \right) \theta. \] (2.24)

Then, Lemma 2.1 enables us to obtain the solutions \( D_l \) by solving the following overdetermined system of
where

\[
\int_{-1}^{1} f(t) dt \approx \int_{-1}^{1} \hat{f}(t) dt = \sum_{i=1}^{r} \omega_i f(t_i),
\]  

(2.34)

where the locations of nodes \( t_i \in [-1,1] \) and positive weights \( \omega_i \) are provided in textbooks [1]. For Eq. (2.30), a point \( P_i \) located at the \( j \)-th node of \( Z_{k}Z_{k+1} \in AB \) has the weights

\[
\alpha_i = \sqrt{\omega_i} h_{k}^{1/2}.
\]

The weights \( \beta_i \) and \( \gamma_i \) can be obtained similarly. When \( r = 1 \), the Gaussian rule is just the central rule with \( t_1 = 0 \) and \( \omega_1 = 2 \). For the Gaussian rules, we have the following proposition, similar to Lemma 2.1.

**Proposition 2.1.** Let the Gaussian rules (2.34) be used in Eq. (2.16), and \( m \geq N + 1 \). Then the coefficients \( D_{\varepsilon} \) from the corresponding collocation equations (2.30)–(2.32) are just the solutions from Eq. (2.15).

Let us consider the computer complexity of this method.

In Eq. (2.29) we may employ the recursive formulas to save CPU time:

\[
\begin{align*}
F_{i,l} &= \begin{cases} 
\sqrt{h_{i}^{l+1/2}} \cos \left( l + \frac{1}{2} \right) \theta_i, & 1 \leq i \leq M, 0 \leq l \leq N, \\
\sqrt{h_{i}^{l-1/2}} \cos \left( l - \frac{1}{2} \right) \theta_{i-M}, & M < i \leq 2M, 0 \leq l \leq N, \\
\sqrt{h_{i}^{l-1/2}} \sin \left( l - \frac{1}{2} \right) \theta_{i-2M}, & 2M < i \leq 3M, 0 \leq l \leq N, \\
\sqrt{h_{i}^{l-1/2}} \sin \left( l - \frac{1}{2} \right) \theta_{i-3M}, & 3M < i \leq 4M, 0 \leq l \leq N.
\end{cases}
\end{align*}
\]  

(2.29)

In general, we can rewrite the overdetermined system of Eqs. (2.25)–(2.27) as

\[
\alpha_i (u_N(P_i) - 500) = 0, \quad P_i \in \bar{AB},
\]  

(2.30)

\[
w_\beta_i \frac{\partial u_N}{\partial x} (P_i^l) = 0, \quad P_i^l \in CD,
\]  

(2.31)

\[
w_\gamma \frac{\partial u_N}{\partial y} (Q_i) = 0, \quad Q_i \in BC,
\]  

(2.32)

where \( P_i \) and \( P_i^l \) and \( Q_i \) are the nodes of integration rules, and \( \alpha_i \), \( \beta_i \) and \( \gamma_i \) are positive weights. Eqs. (2.30)–(2.32) may come from other quadratures. Take the Gaussian rules for example. Denote \( h_k = Z_{k}Z_{k+1}/2 \). By using an affine transformation, the interval \( [Z_{k}, Z_{k+1}] \) can be converted to \([-1,1]\). Hence by this transformation, \( x \in [Z_{k}Z_{k+1}] \) is mapped to \( t \in [-1,1] \), \( f(x) \) to \( \hat{f}(t) \), and the integral on \( [Z_{k}Z_{k+1}] \) is changed to

\[
\int_{Z_{k}Z_{k+1}} f(x) dx = \frac{h_k}{2} \int_{-1}^{1} \hat{f}(t) dt.
\]  

(2.33)

In fact, the entries of the overdetermined system of Eqs. (2.25)–(2.27) may come from other quadratures. Take

\[
F_\xi = \begin{bmatrix} F_1, & F_2, & \cdots, & F_m \end{bmatrix},
\]  

(2.28)

where the associated matrix \( F \in R^{\infty(N+1)} \), \( \bar{b}^* \in R^m \) and \( \bar{\xi} \in R^{N+1} \). In fact, the entries of \( F = (F_{i,l}) \) are given by

\[
\begin{align*}
D_{r}^{l+1/2} &= \sqrt{h_{r}^{l+1/2}} \cos \left( l + \frac{1}{2} \right) \theta_r, \\
D_{r-1}^{l-1/2} &= \sqrt{h_{r}^{l-1/2}} \cos \left( l - \frac{1}{2} \right) \theta_{r-M}, \\
D_{r-2}^{l-1/2} &= \sqrt{h_{r}^{l-1/2}} \sin \left( l - \frac{1}{2} \right) \theta_{r-2M}, \\
D_{r-3}^{l-1/2} &= \sqrt{h_{r}^{l-1/2}} \sin \left( l - \frac{1}{2} \right) \theta_{r-3M},
\end{align*}
\]  

\[
\theta_r = \pi - \theta_r, \quad 1 \leq i \leq M,
\]  

(2.26)

\[
\begin{align*}
D_{r}^{l+1/2} &= \sqrt{h_{r}^{l+1/2}} \cos \left( l + \frac{1}{2} \right) \theta_0, \\
D_{r-M}^{l-1/2} &= \sqrt{h_{r}^{l-1/2}} \cos \left( l - \frac{1}{2} \right) \theta_{r-M}, \\
D_{r-2M}^{l-1/2} &= \sqrt{h_{r}^{l-1/2}} \sin \left( l - \frac{1}{2} \right) \theta_{r-2M}, \\
D_{r-3M}^{l-1/2} &= \sqrt{h_{r}^{l-1/2}} \sin \left( l - \frac{1}{2} \right) \theta_{r-3M},
\end{align*}
\]  

\[
\theta_0 = \pi - \theta_r, \quad 1 \leq i \leq M,
\]  

(2.27)

where \( m = 4M > N + 1 \). Denote the overdetermined system of Eqs. (2.25)–(2.27) by

\[
\begin{align*}
\cos \left( l + \frac{1}{2} \right) \theta_r &= 2 \cos \theta_r \cos \left( l - \frac{1}{2} \right) \theta_r - \cos \left( l - \frac{3}{2} \right) \theta_r, \\
r_q^{l+1/2} &= r_q^{l-1/2},
\end{align*}
\]  

(2.35)

To solve the least squares solution of Eq. (2.28) with full rank \( F \), we may use the QR method by the Householder orthogonalization with the flops [7, p. 248]

\[
T_{k} = 2mn^{2} - \frac{2n^{3}}{3}, \quad n = N + 1.
\]  

(2.36)

On the other hand, the normal equations from Eq. (2.28) are

\[
\tilde{A} = F^{T}F = F^{T}\bar{b} = \bar{b},
\]  

(2.37)

where \( A \) is symmetric positive definite. Then the flops for \( F^{T}F \) and the Gaussian elimination of symmetric matrices are \( m(n^{2}+n) \) and \((1/3)n^{3}\), respectively. So the main flops needed is

\[
T_{N} = mn^{2} + \frac{1}{3}n^{3}.
\]  

(2.38)
where solutions and condition numbers are listed in Tables 1 and 2, for Motz’s problem. First, for the central rule, errors of the Trefftz method has been explored in Ref. [14].

Communication with others.

The error norms and condition numbers from the collocation Trefftz method for Motz’s problem by the central rule obtained by the algorithms described in this paper using the central rule.

Table 1

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>$|e|_B$</th>
<th>$|e|_{\infty,TB}$</th>
<th>Cond.</th>
<th>$|\Delta D_0|_{D_0}$</th>
<th>$|\Delta D_1|_{D_1}$</th>
<th>$|\Delta D_2|_{D_2}$</th>
<th>$|\Delta D_3|_{D_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8</td>
<td>0.250(−1)</td>
<td>0.149(−1)</td>
<td>94.3</td>
<td>0.189(−5)</td>
<td>0.491(−5)</td>
<td>0.601(−5)</td>
<td>0.928(−3)</td>
</tr>
<tr>
<td>18</td>
<td>12</td>
<td>0.133(−3)</td>
<td>0.811(−4)</td>
<td>0.193(4)</td>
<td>0.158(−7)</td>
<td>0.113(−6)</td>
<td>0.290(−6)</td>
<td>0.502(−6)</td>
</tr>
<tr>
<td>26</td>
<td>16</td>
<td>0.973(−6)</td>
<td>0.734(−6)</td>
<td>0.366(5)</td>
<td>0.216(−9)</td>
<td>0.155(−8)</td>
<td>0.380(−8)</td>
<td>0.202(−8)</td>
</tr>
<tr>
<td>34</td>
<td>24</td>
<td>0.839(−8)</td>
<td>0.459(−8)</td>
<td>0.666(6)</td>
<td>0.160(−11)</td>
<td>0.121(−10)</td>
<td>0.296(−10)</td>
<td>0.152(−10)</td>
</tr>
</tbody>
</table>

In our case, $n = N + 1$ and $m = 4N$. Evidently, when $m \gg n$, we have $T_L \approx 2T_N$, and when $m \approx n$

\[ T_L - T_N = (m - n)n^2 \geq 0. \]

Then we conclude the following

**Corollary 2.1.** The flops needed to solve a least squares problem (2.28) by the Householder QR method are larger than, but at most double of those by the normal method (2.37).

Besides the QR method, the singular value decomposition (SVD) can also be used to solve the overdetermined system (2.28). A comparison between the QR method and the SVD is given in Ref. [3]. In general, the latter uses more flops than the former. So the SVD is not recommended here.

Since the condition number of matrix $A$ is nearly square of that of matrix $F$ [7], using the normal equations incurs a serious loss of solution accuracy for Motz’s problem. Some numerical experiments of the normal method were reported in Lefeber [12], where only four and five significant digits of Motz’s solutions were obtained from the computation in double precision. Hence to obtain the numerical solutions of Motz’s problem, we always choose the QR method to solve Eq. (2.28). Such a numerical approach is called the BAM in Refs. [14,16] and the collocation Trefftz method in this paper.\(^1\) Note that stability analysis of the collocation Trefftz method has been explored in Ref. [14].

To close this section, we provide numerical experiments for Motz’s problem. First, for the central rule, errors of the solutions and condition numbers are listed in Tables 1 and 2, where $e = u - u_N$, $M$ denotes the number of collocation nodes along $AB$, and the total number of all collocation nodes used is $4M$. In these tables, $\Delta D_j = d_j - D_j$, $\|e\|_{\infty,TB} = \max_{TB} |e|$, and the condition number is defined by

\[ \text{Cond.} = \left\{ \begin{array}{c} \lambda_{\max}(F^TF) \\ \lambda_{\min}(F^TF) \end{array} \right\}^{1/2} = \left\{ \begin{array}{c} \lambda_{\max}(A) \\ \lambda_{\min}(A) \end{array} \right\}^{1/2}, \]

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the maximal and minimal eigenvalues of $A$, respectively. It can be seen from Table 2 that $M$ should be chosen as $M \approx N/2$ for $N = 34$.

**Tables 1–16** are all computed by means of Fortran programs in double precision.

Moreover, for the Gaussian rule with six nodes and those with 1, 2, 4, 6, 8 and 10 nodes, the results are listed in Tables 3 and 4, respectively, and the best leading coefficients in Table 5 by the Gaussian rule with six nodes as $N = 34$ and $M = 30$. Note that the central rule is the simplest Gaussian rule with $r = 1$. When using the Gaussian rule, there seems no significant effect to reduce the errors $\|e\|_B$ and $\|e\|_{\infty,TB}$ (e.g. from $\|e\|_B = 0.839(−8)$ down to 0.428(−8), Table 4). From Table 4, however, the Gaussian rules of high order do improve evidently the accuracy of leading coefficients. For $N = 34$, $M = 30$, and the Gaussian rule of six nodes, the highly accurate solutions are listed in Table 5 with the best leading coefficient $D_0 = 401.162453745234416$.

\[ (2.39) \]

Compared this $D_0$ in Eq. (2.39) with more accurate values [14,15] using Mathematica, the relative error is less than the rounding error of double precision.\(^2\) Note that $D_0$ in Eq. (2.39) has 17 significant decimal digits; while the $D_0$ in Refs. [14,16] has only 12 significant digits. This is an important evolution of Refs. [14,16]. Besides, we also list $D_j$ with significant digits (Sig. digits) in Table 5, which are obtained from $D_j$ with all digits by rounding. The errors of Sig digits occur only at the last digit at most with a half unit, compared with the more accurate coefficients in Ref. [15]. Although $D_{28} - D_{34}$ are incorrect, they are indispensable to reach the global optimal solutions. Hence, the solutions from this paper are optimal in the global errors,

\[ 2 \text{ This seems impossible! In fact, there exist some guard digits in computer for arithmetic of floating point numbers by noting that there are 18 digits in computer outputs of double precision, and some cancellation of rounding errors in statistics may also happen in the computation. Hence, this occasionally excellent results may happen in random, which have been caught carefully by our computation and provided in Tables 5, 12 and 15. However, we can see from Table 4 that coefficient } D_0 \text{ has at least 16 significant digits by the Gaussian rule with high order.} \]
and the highly accurate leading coefficients are natural consequences.

3. Error bounds and integration approximation

Define the norm

\[ \| v \|_{0,AB} = \left( \int_{AB} v^2 \, ds \right)^{1/2}. \]

Then for any \( w > 0 \), there exists a constant \( C \) independent of \( N \) and \( u \) such that

\[ \| u - u_N \|_1 \leq C \left( K_N + \frac{1}{w} \right) \| u - u_N \|_B. \]

We cite a lemma from Refs. [14,16].

**Lemma 3.1.** Let \( u \in H^1(S) \) be the solution of Motz’s problem. If the following inverse property holds

\[ \| v \|_{0,AB} \leq K_N \| v \|_1, \quad v \in V_N, \]  

where

\[ \| v \|_1 = \left( \int_S (v_x^2 + v_y^2 + v_z^2) \, ds \right)^{1/2}. \]

Below, new analysis is devoted to the collocation Trefftz method involving numerical approximation of integration.

Table 2
The error norms and condition numbers from the collocation Trefftz method for Motz’s problem by the central rule as \( N = 34 \)

<table>
<thead>
<tr>
<th>( M )</th>
<th>( | e |_R )</th>
<th>( | e |_{\infty,\Omega} )</th>
<th>Cond.</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.135(−8)</td>
<td>0.496(−6)</td>
<td>0.267(8)</td>
<td>0.377(−9)</td>
<td>0.266(−8)</td>
<td>0.641(−8)</td>
<td>0.342(−8)</td>
</tr>
<tr>
<td>12</td>
<td>0.587(−8)</td>
<td>0.713(−7)</td>
<td>0.996(6)</td>
<td>0.337(−10)</td>
<td>0.239(−9)</td>
<td>0.578(−9)</td>
<td>0.305(−9)</td>
</tr>
<tr>
<td>16</td>
<td>0.772(−8)</td>
<td>0.189(−7)</td>
<td>0.679(6)</td>
<td>0.729(−11)</td>
<td>0.520(−10)</td>
<td>0.127(−9)</td>
<td>0.655(−10)</td>
</tr>
<tr>
<td>24</td>
<td>0.839(−8)</td>
<td>0.459(−6)</td>
<td>0.669(6)</td>
<td>0.169(−11)</td>
<td>0.121(−13)</td>
<td>0.296(−10)</td>
<td>0.152(−10)</td>
</tr>
<tr>
<td>32</td>
<td>0.849(−8)</td>
<td>0.462(−6)</td>
<td>0.669(6)</td>
<td>0.769(−11)</td>
<td>0.550(−11)</td>
<td>0.134(−10)</td>
<td>0.695(−11)</td>
</tr>
</tbody>
</table>

| * The errors less than computer rounding errors in double precision.

Table 3
The error norms and condition numbers from the collocation Trefftz method for Motz’s problem as \( N = 34 \) by the Gaussian rule with six nodes

<table>
<thead>
<tr>
<th>( M )</th>
<th>( | e |_R )</th>
<th>( | e |_{\infty,\Omega} )</th>
<th>Cond.</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
<th>( \frac{\Delta D_1}{D_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.359(−8)</td>
<td>0.721(−8)</td>
<td>0.675(6)</td>
<td>0.531(−13)</td>
<td>0.646(−12)</td>
<td>0.405(−11)</td>
<td>0.868(−11)</td>
</tr>
<tr>
<td>18</td>
<td>0.494(−8)</td>
<td>0.629(−8)</td>
<td>0.679(6)</td>
<td>0.468(−14)</td>
<td>0.211(−14)</td>
<td>0.620(−13)</td>
<td>0.352(−14)</td>
</tr>
<tr>
<td>24</td>
<td>0.491(−8)</td>
<td>0.530(−8)</td>
<td>0.679(6)</td>
<td>0.567(−15)</td>
<td>0.324(−15)</td>
<td>0.103(−14)</td>
<td>0.337(−13)</td>
</tr>
<tr>
<td>34</td>
<td>0.493(−8)</td>
<td>0.520(−8)</td>
<td>0.676(6)</td>
<td>0.850(−15)</td>
<td>0.324(−15)</td>
<td>0.103(−14)</td>
<td>0.308(−13)</td>
</tr>
<tr>
<td>36</td>
<td>0.494(−8)</td>
<td>0.520(−8)</td>
<td>0.676(6)</td>
<td>0.850(−15)</td>
<td>0.324(−15)</td>
<td>0.103(−14)</td>
<td>0.308(−13)</td>
</tr>
</tbody>
</table>

* The errors less than computer rounding errors in double precision.
Table 5
The leading coefficients $D_i$ from the collocation Trefftz method for Motz’s problem by the Gaussian rule with six nodes as $N = 34$ and $M = 30$

<table>
<thead>
<tr>
<th>$i$</th>
<th>All digits</th>
<th>Significant digits</th>
<th>Number of significant digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>401.1624537452344416</td>
<td>401.162453745234442</td>
<td>17</td>
</tr>
<tr>
<td>1</td>
<td>87.659201950879299</td>
<td>87.659201950879299</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>17.2379150794467897</td>
<td>17.2379150794467897</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>$-8.071215259687970$</td>
<td>$-8.071215259687970$</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>1.44027271702238968</td>
<td>1.44027271702238970</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>0.331054885920006037</td>
<td>0.331054885920006037</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>0.27543734507860671</td>
<td>0.27543734507860671</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>$-0.869329945041107943(-1)$</td>
<td>$-0.869329945041107945(-1)$</td>
<td>9</td>
</tr>
<tr>
<td>8</td>
<td>0.336048784207248854</td>
<td>0.336048784207248854</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>0.1538437449014131(-1)</td>
<td>0.1538437449014131(-1)</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>0.73023016473715797(-2)</td>
<td>0.73023016473715797(-2)</td>
<td>7</td>
</tr>
<tr>
<td>11</td>
<td>$-0.31841361654662899(-2)$</td>
<td>$-0.31841361654662899(-2)$</td>
<td>7</td>
</tr>
<tr>
<td>12</td>
<td>0.12206458154974736(-2)</td>
<td>0.12206458154974736(-2)</td>
<td>7</td>
</tr>
<tr>
<td>13</td>
<td>0.53096529852850803(-3)</td>
<td>0.53096529852850803(-3)</td>
<td>6</td>
</tr>
<tr>
<td>14</td>
<td>0.271512202889081647(-3)</td>
<td>0.271512202889081647(-3)</td>
<td>6</td>
</tr>
<tr>
<td>15</td>
<td>$-0.12004504373287966(-3)$</td>
<td>$-0.12004504373287966(-3)$</td>
<td>5</td>
</tr>
<tr>
<td>16</td>
<td>0.50538241419159885(-4)</td>
<td>0.50538241419159885(-4)</td>
<td>4</td>
</tr>
<tr>
<td>17</td>
<td>0.231662561135488172(-4)</td>
<td>0.231662561135488172(-4)</td>
<td>4</td>
</tr>
<tr>
<td>18</td>
<td>0.1153486476555884393(-4)</td>
<td>0.1153486476555884393(-4)</td>
<td>5</td>
</tr>
<tr>
<td>19</td>
<td>0.52932380785491411(-5)</td>
<td>0.52932380785491411(-5)</td>
<td>3</td>
</tr>
<tr>
<td>20</td>
<td>0.228975882995898625(-3)</td>
<td>0.228975882995898625(-3)</td>
<td>3</td>
</tr>
<tr>
<td>21</td>
<td>0.106239406374917051(-5)</td>
<td>0.106239406374917051(-5)</td>
<td>3</td>
</tr>
<tr>
<td>22</td>
<td>0.53072526325856923(-6)</td>
<td>0.53072526325856923(-6)</td>
<td>3</td>
</tr>
<tr>
<td>23</td>
<td>$-0.24507485537844696(-6)$</td>
<td>$-0.24507485537844696(-6)$</td>
<td>2</td>
</tr>
<tr>
<td>24</td>
<td>0.1064498329739802(-6)</td>
<td>0.1064498329739802(-6)</td>
<td>2</td>
</tr>
<tr>
<td>25</td>
<td>0.51034714154622412(-7)</td>
<td>0.51034714154622412(-7)</td>
<td>1</td>
</tr>
<tr>
<td>26</td>
<td>0.254090364217598898(-7)</td>
<td>0.254090364217598898(-7)</td>
<td>1</td>
</tr>
<tr>
<td>27</td>
<td>$-0.11046492942198792(-7)$</td>
<td>$-0.11046492942198792(-7)$</td>
<td>1</td>
</tr>
<tr>
<td>28</td>
<td>0.493426255784041917(-8)</td>
<td>0.493426255784041917(-8)</td>
<td>0</td>
</tr>
<tr>
<td>29</td>
<td>0.232829745036186828(-8)</td>
<td>0.232829745036186828(-8)</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>0.115208023942516515(-8)</td>
<td>0.115208023942516515(-8)</td>
<td>0</td>
</tr>
<tr>
<td>31</td>
<td>$-0.345561696019388690(-9)$</td>
<td>$-0.345561696019388690(-9)$</td>
<td>0</td>
</tr>
<tr>
<td>32</td>
<td>0.153086899837533823(-9)</td>
<td>0.153086899837533823(-9)</td>
<td>0</td>
</tr>
<tr>
<td>33</td>
<td>0.72277055419099639(-10)</td>
<td>0.72277055419099639(-10)</td>
<td>0</td>
</tr>
<tr>
<td>34</td>
<td>0.352933030515648864(-10)</td>
<td>0.352933030515648864(-10)</td>
<td>0</td>
</tr>
</tbody>
</table>

Denote

\[
[u, v]_B = \left( \int_{\Gamma_B} uv \, d\ell + w^2 \int_{\Gamma_{C\Omega B}} u v \, d\ell \right)^{1/2}, \tag{3.2}
\]

\[
\|v\|_B = \sqrt{\int_{\Gamma_B} v^2 \, d\ell + w^2 \int_{\Gamma_{C\Omega B}} v^2 \, d\ell}^{1/2}. \tag{3.3}
\]

The solutions $\tilde{u}_N$ of Eq. (2.15) will satisfy

\[
\|u - \tilde{u}_N\|_B = \min_{v \in V_N} \|u - v\|_B = \min_{v \in V_N} \sqrt{I(v)}. \tag{3.4}
\]

For the integration rules involved, we denote

\[
\|v\|_B^2 = \int_{\Gamma_B} v^2 \, d\ell. \tag{3.5}
\]

where $v^2$ are piecewise interpolation polynomials of $v^2$ with order $k$ along $\Gamma = \partial S$. We can prove the following lemma, similarly from Refs. [14,16].

\textbf{Lemma 3.2.} The solutions $\tilde{u}_N$ obtained by the collocation Trefftz methods with integral approximation satisfy

\[
[u - \tilde{u}_N, v]_B = 0, \quad \forall v \in V_N, \tag{3.6}
\]

and

\[
\|v - \tilde{u}_N\|_B = \|v\|_B, \quad \forall v \in V_N. \tag{3.7}
\]

Table 6
The errors and condition numbers from the collocation Trefftz method by the central rule for the cracked beam problem with $u_N$, where $w = 1/(N+1)$

<table>
<thead>
<tr>
<th>$N + 1$</th>
<th>$|u - u_N|_B$</th>
<th>$|u - u_N|_{w B}$</th>
<th>Cond.</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.174(-1)</td>
<td>0.192(-1)</td>
<td>118</td>
</tr>
<tr>
<td>20</td>
<td>0.103(-3)</td>
<td>0.143(-3)</td>
<td>0.242(4)</td>
</tr>
<tr>
<td>28</td>
<td>0.780(-6)</td>
<td>0.126(-5)</td>
<td>0.457(5)</td>
</tr>
<tr>
<td>36</td>
<td>0.697(-8)</td>
<td>0.123(-7)</td>
<td>0.828(6)</td>
</tr>
<tr>
<td>44</td>
<td>0.655(-10)</td>
<td>0.128(-9)</td>
<td>0.148(8)</td>
</tr>
</tbody>
</table>
Next, let us examine the errors from integration rules. Suppose that the rules are chosen to have the following relative errors for $v$ and $u - v$, where $v \in V_N$:
\[
\left| \frac{\left( \int_{AB} - \hat{I}_{AB} \right) v^2 \, d\ell}{\int_{AB} v^2 \, d\ell} \right| \leq b < \frac{3}{4}, \tag{3.8}
\]
\[
\left| \frac{\left( \int_{BC} - \hat{I}_{BC} \right) v^2 \, d\ell}{\int_{BC} v^2 \, d\ell} \right| \leq b < \frac{3}{4}, \tag{3.9}
\]
\[
\left| \frac{\left( \int_{CD} - \hat{I}_{CD} \right) v^2 \, d\ell}{\int_{CD} v^2 \, d\ell} \right| \leq b < \frac{3}{4}. \tag{3.10}
\]

Table 8

The errors and condition numbers from the collocation Trefftz method by the central rule for the cracked beam problem with $u_0^c$ where $w = 1/(N + 1)$.

<table>
<thead>
<tr>
<th>$N + 1$</th>
<th>$|u - u_0^c|_B$</th>
<th>$|u - u_0^c|_B$</th>
<th>Cond.</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>0.181 (-1)</td>
<td>0.143 (-1)</td>
<td>14.7</td>
</tr>
<tr>
<td>20</td>
<td>0.108 (-3)</td>
<td>0.860 (-4)</td>
<td>179</td>
</tr>
<tr>
<td>28</td>
<td>0.835 (-6)</td>
<td>0.673 (-6)</td>
<td>0.24 (4)</td>
</tr>
<tr>
<td>36</td>
<td>0.731 (-8)</td>
<td>0.593 (-8)</td>
<td>0.34 (5)</td>
</tr>
<tr>
<td>44</td>
<td>0.689 (-10)</td>
<td>0.563 (-10)</td>
<td>0.49 (6)</td>
</tr>
</tbody>
</table>

Table 9

The coefficients from the collocation Trefftz method by the central rule for the cracked beam problem with $u_0^c$ where $N = 43$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$D_i$</th>
<th>$i$</th>
<th>$D_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>540.565122713627</td>
<td>22</td>
<td>0.741136835680306 (-12)</td>
</tr>
<tr>
<td>1</td>
<td>-167.041350909274</td>
<td>23</td>
<td>0.417873188876248 (-11)</td>
</tr>
<tr>
<td>2</td>
<td>0.19819874257744 (-13)</td>
<td>24</td>
<td>0.121996855522588 (-7)</td>
</tr>
<tr>
<td>3</td>
<td>-0.21918536582999 (-13)</td>
<td>25</td>
<td>0.143269024396301 (-6)</td>
</tr>
<tr>
<td>4</td>
<td>-2.21801471698044</td>
<td>26</td>
<td>0.853944744811233 (-12)</td>
</tr>
<tr>
<td>5</td>
<td>-1.68233110389621</td>
<td>27</td>
<td>0.392784692715081 (-11)</td>
</tr>
<tr>
<td>6</td>
<td>-0.21465946470374 (-14)</td>
<td>28</td>
<td>0.519522874909789 (-9)</td>
</tr>
<tr>
<td>7</td>
<td>0.97515263890286 (-14)</td>
<td>29</td>
<td>0.71669781571644 (-8)</td>
</tr>
<tr>
<td>8</td>
<td>-0.722712676630922 (-2)</td>
<td>30</td>
<td>0.562344162084752 (-12)</td>
</tr>
<tr>
<td>9</td>
<td>-0.41962077504757 (-11)</td>
<td>31</td>
<td>0.21002519224617 (-11)</td>
</tr>
<tr>
<td>10</td>
<td>0.15836858156426 (-13)</td>
<td>32</td>
<td>0.22612209663434 (-10)</td>
</tr>
<tr>
<td>11</td>
<td>0.37950008821883 (-13)</td>
<td>33</td>
<td>0.36168802767140 (-9)</td>
</tr>
<tr>
<td>12</td>
<td>-0.349003797729518 (-3)</td>
<td>34</td>
<td>0.20994050599741 (-12)</td>
</tr>
<tr>
<td>13</td>
<td>0.154580080052455 (-2)</td>
<td>35</td>
<td>0.631754175553925 (-12)</td>
</tr>
<tr>
<td>14</td>
<td>0.893243717123692 (-13)</td>
<td>36</td>
<td>0.907100573484872 (-12)</td>
</tr>
<tr>
<td>15</td>
<td>0.73345776410472 (-12)</td>
<td>37</td>
<td>0.16631437291397 (-10)</td>
</tr>
<tr>
<td>16</td>
<td>-0.52412646166936 (-5)</td>
<td>38</td>
<td>0.417258876739002 (-13)</td>
</tr>
<tr>
<td>17</td>
<td>-0.649439289221108 (-4)</td>
<td>39</td>
<td>0.988154926645705 (-13)</td>
</tr>
<tr>
<td>18</td>
<td>0.356608390179608 (-12)</td>
<td>40</td>
<td>0.23053504761913 (-13)</td>
</tr>
<tr>
<td>19</td>
<td>0.24490448645645 (-11)</td>
<td>41</td>
<td>0.49326478414809 (-9)</td>
</tr>
<tr>
<td>20</td>
<td>-0.31791558260462 (-6)</td>
<td>42</td>
<td>0.32864945573497 (-14)</td>
</tr>
<tr>
<td>21</td>
<td>-0.29697061602140 (-5)</td>
<td>43</td>
<td>0.62166152740370 (-14)</td>
</tr>
</tbody>
</table>

where $b$ is a constant. Then we have the following proposition.

**Proposition 3.1.** For those rules of quadrature satisfying Eqs. (3.8)–(3.10), the following bound holds
\[
\left| \frac{\|v\|_B - \|v\|_B}{\|v\|_B} \right| \leq a < \frac{1}{2}, \quad v \in V_N, \tag{3.11}
\]
where $a = 1 - \sqrt{1 - b}$ is a constant.

**Proof.** We have from the assumptions
\[
\frac{\|v\|_B^2 - \|v\|_B^2}{\|v\|_B^2} \leq \frac{1}{4} \left( \left( \int_{AB} - \hat{I}_{AB} \right) v^2 \, d\ell + \int_{BC \cup CD} - \hat{I}_{BC \cup CD} \right) v^2 \, d\ell \right| \leq b. \tag{3.12}
\]
We obtain
\[
1 - b \geq \frac{\|v\|_B^2}{\|v\|_B^2} \leq 1 + b.
\]
The above equation gives
\[
\sqrt{1 - b} \leq \frac{\|v\|_B}{\|v\|_B} \leq 1 + b. \tag{3.13}
\]

Next, we have from Eqs. (3.12) and (3.13)
\[
\frac{\|v\|_B - \|v\|_B}{\|v\|_B + \|v\|_B} \leq \frac{b}{1 + \sqrt{1 - b}} = 1 - \sqrt{1 - b} = a < \frac{1}{2}. \tag{3.14}
\]
This completes the proof of Proposition 3.1.
Table 10
The error norms and condition numbers from the collocation Trefftz method for the cracked beam problem as \( N = 43 \) by the Gaussian rule with eight nodes

| \( M \) | \( \|u - \hat{u}\|_B \) | \( \|u - u^0\|_{\infty, \text{Tre}} \) | Cond. | \( \frac{|\Delta D_b|}{D_b} \) | \( \frac{|\Delta D_1|}{D_1} \) | \( \frac{|\Delta D_2|}{D_2} \) | \( \frac{|\Delta D_3|}{D_3} \) |
|---|---|---|---|---|---|---|---|
| 16 | 0.317(−10) | 0.570(−10) | 0.447(6) | 0.421(−15) | 0.340(−15) | 0.340(−14) | 0.647(−14) |
| 24 | 0.319(−10) | 0.527(−10) | 0.447(6) | 0.421(−15) | 0.340(−15) | 0.340(−14) | 0.510(−15) |
| 32 | 0.319(−10) | 0.526(−10) | 0.447(6) | 0.841(−15) | 0.340(−15) | 0.541(−14) | 0.594(−14) |
| 40 | 0.319(−10) | 0.526(−10) | 0.447(6) | 0.631(−15) | 0.510(−15) | 0.801(−15) | 0.792(−14) |

* The errors less than computer rounding errors in double precision.

Take the central rule in Eqs. (2.12) and (2.13) for example. We have from Ref. [1]

\[
\left( \int_{\mathcal{A}B} - \int_{\mathcal{B}A} \right) \, d\ell = \frac{\mu^2}{24} f''(\xi),
\]

where \( f = v^2 \) or \( f = (u - v)^2 \), and \( \xi \in \mathcal{A}B \). Since \( f'' = 2(v'(v')^2 + vv'') \) for \( f = v^2 \), the requirements of quadrature errors in Proposition 3.1 imply that

\[
\frac{1}{4} \int_{\mathcal{A}B} v^2 \, d\ell \leq \frac{3}{4} \int_{\mathcal{A}B} \nu^2 \, d\ell,
\]

or equivalently

\[
\frac{\mu^2}{12} |(v'(v')^2 + vv'')| \leq \frac{3}{4} \int_{\mathcal{A}B} \nu^2 \, d\ell.
\]

Next, we give a new lemma.

**Lemma 3.3.** Suppose that the rules of integrations in Eq. (2.16) are chosen to satisfy the bound (3.11). Then, the norms \( \|\cdot\|_B \) and \( \|\cdot\|_B \) defined in Eqs. (2.11) and (3.3) are equivalent to each other

\[
C_1 \|v\|_B \leq \|v\|_B \leq C_2 \|v\|_B, \quad \nu \in V_N,
\]

where \( C_1 \) and \( C_2 \) are two positive constants independent of \( v \) and \( N \).

**Proof.** We have from Eq. (3.11)

\[
\|v\|_B - \|v\|_B \leq a \|v\|_B,
\]

and then

\[
\|v\|_B \leq \frac{1}{1 - a} \|v\|_B.
\]

Also from Eq. (3.11)

\[
\|v\|_B - \|v\|_B \leq a \|v\|_B,
\]

and then

\[
\|v\|_B \leq (1 + a) \|v\|_B.
\]

Hence, the desired result (3.18) follows from Eqs. (3.19) and (3.20). This completes the proof of Lemma 3.3.

Accordingly, we have a new, important theorem.

**Theorem 3.1.** Let the condition (3.1) hold, and the rules of integrations involved in Eq. (2.16) satisfy Eq. (3.11) for \( v \) and \( u - v \), \( \forall v \in V_N \). Then

\[
\|u - \hat{u}_N\|_1 \leq \inf_{v \in V_N} \|u - v\|_1 + C(K_N + 1/w)\|u - v\|_B,
\]

where \( C \) is a bounded constant independent of \( u \), \( v \) and \( N \). Moreover

\[
\|u - \hat{u}_N\|_1 \leq \|R_N\|_1 + C(K_N + 1/w)\|R_N\|_B,
\]

where

\[
R_N = \sum_{i=N+1}^\infty d_i \rho^{i+1/2} \cos \left( i + \frac{1}{2} \right) \theta_i,
\]

and \( d_i \) are the true expansion coefficients.

Table 11
The error norms and condition numbers from the collocation Trefftz method for the cracked beam problem by different Gaussian rules with \( r \) nodes as \( N = 43 \)

| \( r \) | \( M \) | \( \|u - \hat{u}\|_B \) | \( \|u - u^0\|_{\infty, \text{Tre}} \) | Cond. | \( \frac{|\Delta D_b|}{D_b} \) | \( \frac{|\Delta D_1|}{D_1} \) | \( \frac{|\Delta D_2|}{D_2} \) | \( \frac{|\Delta D_3|}{D_3} \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 24 | 0.614(−10) | 0.102(−9) | 0.434(6) | 0.294(−14) | 0.715(−14) | 0.177(−12) | 0.175(−12) |
| 2 | 24 | 0.623(−10) | 0.602(−9) | 0.446(6) | 0.210(−15) | 0.187(−14) | 0.617(−13) | 0.523(−13) |
| 4 | 24 | 0.448(−10) | 0.519(−10) | 0.446(6) | 0.210(−15) | 0.199(−14) | 0.921(−14) | 0.462(−14) |
| 6 | 24 | 0.367(−10) | 0.576(−10) | 0.447(6) | 0.210(−15) | 0.199(−14) | 0.921(−14) | 0.726(−14) |
| 8 | 24 | 0.319(−10) | 0.527(−10) | 0.447(6) | 0.210(−15) | 0.510(−15) | 0.701(−14) | 0.103(−13) |
| 12 | 24 | 0.261(−10) | 0.524(−10) | 0.447(6) | 0.210(−15) | 0.340(−15) | 0.741(−14) | 0.488(−14) |

* The errors less than computer rounding errors in double precision.
The constant $C$ in this paper is used as a generic, bounded constant; their values may be different in different contexts. Let $\eta = v - \tilde{u}_N$, then $\eta \in V_N$ if $v \in V_N$. In view of Eq. (3.24) and the norm equivalence (3.18)
\[ \|u - \tilde{u}_N\|_1 \leq \|u - v\|_1 + \|\eta\|_B \]
\[ \leq \|u - v\|_1 + C(K_N + 1/w)\|w\|_B \leq \|u - v\|_1 + \frac{C}{1}(K_N + 1/w)\|w\|_B. \]  
(3.25)

From the orthogonal property (3.6) we obtain
\[ \|\eta\|_B^2 = [\eta, \eta]_B = [v - u, \eta]_B \leq \|u - v\|\|\eta\|_B. \]
(3.26)

The above bound and the norm equivalence for $u - v$ leads to
\[ \|\eta\|_B \leq \|u - v\|_B \leq C\|u - v\|_B. \]
(3.27)

Combining Eqs. (3.25) and (3.27) gives the first desired result (3.21).
Next, the solution (1.4) with the true coefficients $d_i$ can be split into
\[ u = \tilde{u}_N + R_N, \]
(3.28)

where
\[ \tilde{u}_N = \sum_{i=0}^N d_i r^{i+(1/2)} \cos \left(i + \frac{1}{2}\right) \theta. \]
(3.29)

and the remainder $R_N$ is given by Eq. (3.23). Then let $v = \tilde{u}_N$ in Eq. (3.21) we obtain
\[ \|u - \tilde{u}_N\|_1 \leq \|u - v\|_1 + C(K_N + 1/w)\|u - \tilde{u}_N\|_B \leq \|R_N\|_1 + C(K_N + 1/w)\|R_N\|_B. \]
(3.30)

### Table 12

The leading coefficients from the collocation Trefftz method for the cracked beam problem by the Gaussian rule with eight nodes as $N = 43$ and $M = 24$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\tilde{d}_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>540.56512213627488338</td>
</tr>
<tr>
<td>1</td>
<td>-167.041350909274314063</td>
</tr>
<tr>
<td>4</td>
<td>-2.21801471698042096392</td>
</tr>
<tr>
<td>5</td>
<td>-1.6823311038623896630</td>
</tr>
<tr>
<td>8</td>
<td>-0.722712676629304936332(-2)</td>
</tr>
<tr>
<td>9</td>
<td>-0.41961202077540987910815(-1)</td>
</tr>
<tr>
<td>12</td>
<td>-0.349003797752273547863(-3)</td>
</tr>
<tr>
<td>13</td>
<td>-0.1543800807323280442(-2)</td>
</tr>
<tr>
<td>16</td>
<td>-0.824172472675852766202(-5)</td>
</tr>
<tr>
<td>17</td>
<td>-0.64951207503007010942(-4)</td>
</tr>
<tr>
<td>20</td>
<td>-0.31719568270962735950(-6)</td>
</tr>
<tr>
<td>24</td>
<td>-0.29697080406575792138(-5)</td>
</tr>
<tr>
<td>27</td>
<td>-0.122000284043979050718(-7)</td>
</tr>
<tr>
<td>28</td>
<td>-0.143271318148128170607(-6)</td>
</tr>
<tr>
<td>29</td>
<td>-0.51973784350238683567(-9)</td>
</tr>
<tr>
<td>30</td>
<td>-0.71862511706519452695(-4)</td>
</tr>
<tr>
<td>32</td>
<td>-0.22691618031938157640(-10)</td>
</tr>
<tr>
<td>33</td>
<td>-0.36210903096586822496(-9)</td>
</tr>
<tr>
<td>36</td>
<td>-0.9255363181726153988(-12)</td>
</tr>
<tr>
<td>37</td>
<td>-0.16702774414268219372(-10)</td>
</tr>
<tr>
<td>40</td>
<td>-0.24236652419779362680(-13)</td>
</tr>
<tr>
<td>41</td>
<td>-0.4980657013283416745(-12)</td>
</tr>
</tbody>
</table>

**Proof.** From Refs. [14,16], we have
\[ \|v\|_1 \leq C(K_N + 1/w)\|w\|_B, \quad \forall v \in V_N. \]  
(3.24)

The constant $C$ in this case is used as a generic, bounded constant; their values may be different in different contexts. The above bound and the norm equivalence for $u - v$ leads to
\[ \|\eta\|_B \leq \|u - v\|_B \leq C\|u - v\|_B. \]
(3.27)

Combining Eqs. (3.25) and (3.27) gives the first desired result (3.21).

### Table 13

The leading coefficients $a_i$ from Table 12 by Eq. (5.5) for $a = 1/2$ and $b = .0125$ in the scaled cracked beam problem.
This is the second bound (3.22) as desired, which completes the proof of Theorem 3.1.

Even for the simplest central rule, the relative errors of its approximate integrals has no difficult to be less than three quarters. So the conditions (3.8)–(3.11) can be satisfied easily. Hence, the solutions \( \tilde{u}_N \) may still have the exponential convergence rates. More explanation will be given in Section 5. This is a significant difference from the traditional role of integration in the finite element analysis.

Besides, from Theorem 3.1, there is not much difference between lower-order and higher-order quadratures. However, for the accuracy of the leading coefficient \( D_0 \), the high-order rules, such as the Gaussian quadratures with six and eight nodes, may raise its accuracy, based on Tables 3 and 4. Note that the new analysis of quadratures in this section provides a theoretical foundation for the high accuracy of the collocation Trefftz method.

4. The cracked beam problem

As a variant of Motz’s problem, the cracked beam problem is discussed here. Its highly accurate solution can be sought similarly by the collocation Trefftz method (e.g. the BAM in Refs. [14,16]). Not only its highly accurate solutions are obtained in this paper, but also the highly accurate leading coefficient in double precision can be achieved by the Gaussian rule. Half of its expansion coefficients are zero, which is supported by a posterior analysis. Hence, as a singularity model, the cracked beam problem given in this section seems to be superior to Motz’s problem in Sections 2 and 3.

### Table 14

The error norms and condition numbers from the collocation Trefftz method directly for the traditional cracked beam problem in Fig. 3. The traditional cracked beam problem in \( \tilde{S} \) by different Gaussian rules with different nodes as \( N = 43 \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( M )</th>
<th>( |w - w_N|_{0,\tilde{S}} )</th>
<th>( |w - w_N|_{\infty,\tilde{S}} )</th>
<th>Cond.</th>
<th>( \Delta D_0 )</th>
<th>( \Delta D_1 )</th>
<th>( \Delta D_2 )</th>
<th>( \Delta D_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24</td>
<td>0.180(−13)</td>
<td>0.159(−13)</td>
<td>0.191(10)</td>
<td>0.203(−14)</td>
<td>0.493(−14)</td>
<td>0.142(−12)</td>
<td>0.140(−12)</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
<td>0.160(−13)</td>
<td>0.141(−13)</td>
<td>0.191(10)</td>
<td>0.102(−14)</td>
<td>0.940(−15)</td>
<td>0.124(−13)</td>
<td>0.140(−13)</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
<td>0.116(−13)</td>
<td>0.141(−13)</td>
<td>0.187(10)</td>
<td>0.102(−14)</td>
<td>0.117(−15)</td>
<td>0.111(−14)</td>
<td>0.729(−15)</td>
</tr>
<tr>
<td>6</td>
<td>24</td>
<td>0.955(−14)</td>
<td>0.144(−13)</td>
<td>0.186(10)</td>
<td>0.102(−14)</td>
<td>0.117(−15)</td>
<td>0.111(−14)</td>
<td>0.729(−15)</td>
</tr>
<tr>
<td>8</td>
<td>24</td>
<td>0.833(−14)</td>
<td>0.143(−13)</td>
<td>0.185(10)</td>
<td>0.290(−15)</td>
<td>0.235(−15)</td>
<td>0.166(−14)</td>
<td>0.365(−14)</td>
</tr>
<tr>
<td>12</td>
<td>24</td>
<td>0.680(−14)</td>
<td>0.143(−13)</td>
<td>0.185(10)</td>
<td>0.145(−15)</td>
<td>0.117(−15)</td>
<td>0.138(−15)</td>
<td>0.419(−14)</td>
</tr>
</tbody>
</table>

* The errors less than computer rounding errors in double precision.

### Table 15

The leading coefficients \( \alpha_i \) from the collocation Trefftz method directly from the traditional cracked beam problem in Fig. 3 by the Gaussian rule with six nodes as \( N = 43 \) and \( M = 24 \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>All digits</th>
<th>Significant digits</th>
<th>Number of significant digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.19111863197187209344</td>
<td>0.19111863197187209</td>
<td>17</td>
</tr>
<tr>
<td>1</td>
<td>−0.11811607196650948835</td>
<td>0.1181160719665095</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>−0.12546985977187430753(−1)</td>
<td>0.125469859771874(−1)</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>−0.1903340370825710513(−1)</td>
<td>0.190334037082571(−1)</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>−0.65412484415349167181(−3)</td>
<td>0.65412484415349(−3)</td>
<td>11</td>
</tr>
<tr>
<td>9</td>
<td>−0.7595934797958023040594(−2)</td>
<td>−0.759593479795802304059(−2)</td>
<td>12</td>
</tr>
<tr>
<td>12</td>
<td>−0.50541148575806630688(−3)</td>
<td>−0.50541148575806609321(−3)</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>−0.44771152664487571153(−2)</td>
<td>−0.447711527(−2)</td>
<td>9</td>
</tr>
<tr>
<td>16</td>
<td>−0.19096672465493324123126(−3)</td>
<td>−0.1909647(−3)</td>
<td>7</td>
</tr>
<tr>
<td>17</td>
<td>−0.3009035723224123126(−2)</td>
<td>−0.3009004(−2)</td>
<td>7</td>
</tr>
<tr>
<td>20</td>
<td>−0.1178599364511535464920(−3)</td>
<td>−0.11786(−3)</td>
<td>5</td>
</tr>
<tr>
<td>21</td>
<td>−0.2201904691690683645(−2)</td>
<td>−0.22019(−2)</td>
<td>5</td>
</tr>
<tr>
<td>24</td>
<td>−0.72362052299536040750(−4)</td>
<td>−0.724(−4)</td>
<td>4</td>
</tr>
<tr>
<td>25</td>
<td>−0.169965180428740376094(−2)</td>
<td>−0.170(−2)</td>
<td>4</td>
</tr>
<tr>
<td>28</td>
<td>−0.492760129615704093459(−4)</td>
<td>−0.49(−4)</td>
<td>2</td>
</tr>
<tr>
<td>29</td>
<td>−0.13604326809274110108(−2)</td>
<td>−0.136(−2)</td>
<td>3</td>
</tr>
<tr>
<td>32</td>
<td>−0.34117051733278368583(−4)</td>
<td>0.3(−4)</td>
<td>1</td>
</tr>
<tr>
<td>33</td>
<td>−0.109855994513878601453(−2)</td>
<td>0.1(−2)</td>
<td>1</td>
</tr>
<tr>
<td>36</td>
<td>−0.212639663103781256836(−4)</td>
<td>/</td>
<td>0</td>
</tr>
<tr>
<td>37</td>
<td>−0.80793318272657111506(−5)</td>
<td>/</td>
<td>0</td>
</tr>
<tr>
<td>40</td>
<td>−0.78999544027442427633(−5)</td>
<td>/</td>
<td>0</td>
</tr>
<tr>
<td>41</td>
<td>−0.38260528915432711005(−3)</td>
<td>/</td>
<td>0</td>
</tr>
</tbody>
</table>
Comparisons of the error norms and condition numbers from Table 11 by Eq. (5.9) and directly from the traditional cracked beam problem in Fig. 3 as $N = 43$ and $M = 24$

<table>
<thead>
<tr>
<th>$r$ Nodes</th>
<th>From Table 11 by Eq. (5.9)</th>
<th>Direct computation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(b500)</td>
<td>u - u_N</td>
</tr>
<tr>
<td>1</td>
<td>$0.154(-14)$</td>
<td>$0.434(6)$</td>
</tr>
<tr>
<td>2</td>
<td>$0.156(-14)$</td>
<td>$0.446(6)$</td>
</tr>
<tr>
<td>4</td>
<td>$0.112(-14)$</td>
<td>$0.446(6)$</td>
</tr>
<tr>
<td>6</td>
<td>$0.918(-15)$</td>
<td>$0.447(6)$</td>
</tr>
<tr>
<td>8</td>
<td>$0.798(-15)$</td>
<td>$0.447(6)$</td>
</tr>
<tr>
<td>12</td>
<td>$0.653(-15)$</td>
<td>$0.447(6)$</td>
</tr>
</tbody>
</table>

When the boundary conditions on $AB$ and on $BC$ in Fig. 1 are exchanged as (Fig. 2)

$$u_{AB} = 500, \quad u_{CD} = 0, \quad u_{OD} = 0, \quad u_{AB,CD} = 0,$$

this Laplace boundary value problem gives the cracked beam problem. Its original model in Refs. [4–6,19,21] was defined on the domain

$$\tilde{S} = \{(x,y) - \frac{1}{2} \leq x \leq \frac{1}{2}, 0 \leq y \leq \frac{1}{2}\}$$

in Fig. 3. However, two models in $S$ and $\tilde{S}$ have the same nature. In fact, their solutions can be scaled from one to the other, which will be explained in Section 5. Since function (1.4) is also the solutions of the cracked beam problem, we choose

$$u_N(r, \theta) = \sum_{i=0}^{N} D_i r^{i+(1/2)} \cos \left(i + \frac{1}{2}\right) \theta,$$

where the notations $D_i$ with a hat on its head are used to distinguish with those $D_i$ of Motz’s problem in Section 2. We also use $V_N$ as the finite collection of function (4.2). Since $u_N$ satisfies the Laplace equation in $S$ and the boundary conditions on $OD$ and $OA$ already, the coefficients $D_i$ should be chosen to satisfy the rest boundary conditions as best as possible. Define the error norm on $AB \cup BC \cup CD$

$$\|u - v\|_B = \left\{ \int_{BC} (v - 500)^2 + w^2 \int_{AB \cup CD} v^2 \right\}^{1/2},$$

$$w = \frac{1}{N+1}.$$

The solution $u_N$ can be obtained by

$$\|u - u_N\|_B = \inf_{v \in V_N} \|u - v\|_B,$$

where

$$\|v\|_B = \left\{ \int_{BC} v^2 + w^2 \int_{AB \cup CD} v^2 \right\}^{1/2}.$$

We first employ the central rule with a uniform distributed points $P_i$ on $AB \cup BC \cup CD$. We may require $\sqrt{N} = \sqrt{500}$ at $P_i \in BC$ and $\sqrt{N\mu} = 0$ at $P_i \notin AB \cup CD$. Let the number $4M$ of all collocation nodes $P_i$ be larger than $N + 1$, then we obtain an overdetermined system of linear algebraic equations $F \vec{x} = \vec{b}$, where $F$ is a matrix of $4M \times (N + 1)$, and $\vec{x}$ is the unknown vector consisting of $\vec{D}$. We employ the LSM in Section 2 to solve it. The errors, condition numbers and the leading coefficients are given in Tables 6 and 7. It is interesting from Table 7 to note that $D_{4k+2} \approx D_{4k+3} \approx 0$. Hence, we may simply seek a solution of the following simplified forms

$$u_N^* = \sum_{\xi=0}^{L} \sum_{k=0}^{N} D_{4k+4} \xi \cos \left(4\xi + k + \frac{1}{2}\right) \theta,$$

where $N = 4L + 1$. Denote by $V_N^*$ the finite collection of functions in Eq. (4.6). Hence another collocation Trefftz method can be formulated as in Section 2: to seek the solution $u_N^* \in V_N^*$ such that

$$\|u - u_N^*\|_B = \inf_{\vec{v} \in V_N^*} \|u - \vec{v}\|_B,$$

where $\|\vec{v}\|_B$ is defined in Eq. (4.5). Its results are given in Tables 8 and 9. From Tables 6 and 8, we have observed the asymptotes:

$$\|u - u_N\|_B = O(0.553^N), \quad \|u - u_N^*\|_{s,BC} = O(0.564^N), \quad \text{Cond.} = O(1.42^N).$$
\[ u - u^*_N = O(0.558^N), \quad \|u - u^*_N\|_{\infty, \mathcal{T}} = O(0.558^N), \]
Cond. = O(1.39^N). \tag{4.9}

Note that the convergence rates and the condition numbers in Eq. (4.9) are close to those in Eq. (4.8), but only half coefficients of \( u_N \) in Eq. (4.2) are needed. Hence, for the computational purpose, the solution (4.6) with Tables 8 and 9 may be better chosen. From this point of view, the cracked beam using Eq. (4.6) may serve as a better testing model of singularity problems than Motz’s problem.

Compared with the more accurate solutions from CTM [22] using Mathematica with more working digits, the leading coefficients \( \hat{D}_0 \) and \( \hat{D}_1 \) in Table 9 have 15 significant digits.

The analysis in Section 3 can be similarly applied to the collocation Trefftz method for the cracked beam problem. To confirm the admissible functions as Eq. (4.6), we only prove the following proposition.

**Proposition 4.1.** Let the errors \( e_N = u - u^*_N, \) \( N = 4L + 1 \) and
\[
\|e_N\|_{\mathcal{T}} \leq K_N\|e_N\|_{1,S}, \tag{4.10}
\]
where the constant \( K_N \) (\( \geq 1 \)) may be unbounded as \( N \to \infty \).

Suppose
\[
\left( K_N + \frac{1}{w} \right)\|e_N\|_B \to 0, \quad \text{as } N \to \infty. \tag{4.11}
\]
Then the solution of the cracked beam problem can be expressed by
\[
u = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \hat{D}_{4\ell+k} e^{4\ell+k+1/2} \cos \left( 4\ell + k + \frac{1}{2} \right) \theta. \tag{4.12}
\]

**Proof.** From the bounds similar to Lemma 3.1, we have
\[
\|e_N\|_{1,S} = \|u - u^*_N\|_{1,S} \leq C \left( K_N + \frac{1}{w} \right)\|e_N\|_B, \tag{4.13}
\]
where \( C \) is a bounded constant independent of \( N \). From Eqs. (4.11) and (4.13), \( \{e_N\} \) is a bounded sequence. Based on the Kandrasov or Rellich theorem [2], any bounded sequence in the space \( H^1(S) \) contains a subsequence that converges in \( H^1(S) \). Then there must exist a subsequence \( \{e_N^n\} \) in \( H^1(S) \) such that \( \lim_{n \to \infty} e_N^n = \bar{e} \). Since \( \{e_N^n\} \) are bounded in \( H^1(S) \), the convergent limit \( \bar{e} \in H^1(S) \). This implies that
\[
\lim_{N \to \infty} u_N^+ = \lim_{N \to \infty} (u - e_N) = u - \bar{e} = \bar{u} \in H^1(S). \tag{4.11}
\]

Moreover, since \( K_N \geq 1 \) and \( w = 1/(N + 1) \), we conclude that \( \|u - 5000\|_{0, \mathcal{T}} = 0 \) and \( \|\bar{u}\|_{0, \mathcal{T}, \mathbb{B}} = 0 \). Hence, \( \bar{u} \) must be the unique solution of the cracked beam problem. Obviously, the entire sequence \( u^*_N \) also converges to \( \bar{u}(=u) \) based on \( \|u - u^*_N\|_B \to 0 \) as \( N \to \infty \) from Eqs. (4.11) and (4.13). This completes the proof of Proposition 4.1. □

When \( w = 1/(N + 1) \), the empirical exponential convergent rates in Eq. (4.9) guarantee Eq. (4.11). The analysis of Proposition 4.1 is made, based on the a posteriori numerical results, so we call it a posteriori analysis. Proposition 4.1 implies that \( \hat{D}_{4\ell+k} = \hat{D}_{4\ell+k+3} = 0, \forall \ell \geq 0 \).

We also note that condition (4.11) is stronger than that in Table 12.

The errors \( \|u - u^*_N\|_B \) decrease nearly a half, from 0.614(−10) with \( r = 1 \) down to 0.319(−10) with \( r = 8 \). For \( N = 43 \) and \( M = 24 \), the leading coefficients \( \hat{D}_{4\ell+k} \), \( k = 0 \) obtained by the Gaussian rule of eight nodes are reported in Table 12. Compared with the more accurate results in Ref. [22], the relative errors of
\[
\hat{D}_0 = 540.565122713627488338,
\]
from Table 12 has 17 significant digits, and \( \hat{D}_1 \) has 16 significant digits.

5. Discussions and comparisons

Let us consider the cracked beam problem on a scaled domain, \( \tilde{S} = \{(\xi, \eta)| -a < \xi < a, 0 < \eta < a\} \), where the parameter satisfies \( 0 < a \leq 1 \). The scaled cracked beam problem is described by the Laplace equation \( \Delta w = 0 \) on \( \tilde{S} \) satisfying the following boundary conditions
\[
w(\xi, a) = b, \quad -a < \xi < a, \tag{5.1}
\]
\[
w(\xi, 0) = 0, \quad -a < \xi < 0, \quad \frac{\partial w}{\partial \nu}(\xi, 0) = 0, \tag{5.2}
\]
\[0 < \xi < a,\]
\[
\frac{\partial w}{\partial \nu}(\pm a, \eta) = 0, \quad 0 < \eta < a, \quad (5.3)
\]

where \(b\) is a constant, and \(\nu\) is the outward normal to \(\partial S\). Here, another Cartesian coordinate system \((\xi, \eta)\) is chosen. For Fig. 2, \(a = 1\) and \(b = 500\), and for Fig. 3 from the traditional model \([4-6, 19, 21]\), \(a = 1/2\) and \(b = 0.125\). The Laplace solution satisfying Eqs. (5.1)–(5.3) can also be expressed by

\[
w(\xi, \eta) = \sum_{i=0}^{\infty} \alpha_i \rho^{i+1/2} \cos(i + \frac{1}{2}) \theta, \quad (5.4)
\]

where \(\alpha_i\) are the coefficients, \((\rho, \theta)\) are the polar coordinates at the origin \(o\), and \(\rho = \sqrt{\xi^2 + \eta^2}\). There exist the relations for the coefficients of \(\tilde{D}_i\) in Table 12 and \(\alpha_i\):

\[
\alpha_i = \frac{b}{500} \phi_i^{-(i+1/2)}\tilde{D}_i. \quad (5.5)
\]

Now, let us prove Eq. (5.5). Under the affine transformation \(T: (x, y) \rightarrow (\xi, \eta)\), where \(\xi = ax\) and \(\eta = ay\), domain \(S\) is converted to \(\tilde{S}\), and the boundary conditions (4.1) are transformed to

\[
u(\xi, a) = b, \quad -a < \xi < a, \quad (5.6)
\]

\[
u(\xi, 0) = 0, \quad -a < \xi < 0, \quad \frac{\partial u}{\partial \nu}(\xi, 0) = 0, \quad (5.7)
\]

\[
0 < \xi < a,
\]

\[
\frac{\partial u}{\partial \nu}(\pm a, \eta) = 0, \quad 0 < \eta < a. \quad (5.8)
\]

Comparing Eq. (5.6) with \(u_{\text{ref}} = 500\) in Eq. (4.1), we find the relations between \(w\) and \(u\),

\[
w = \frac{b}{500} u. \quad (5.9)
\]

This gives

\[
\sum_{i=0}^{\infty} \alpha_i \rho^{i+1/2} \cos(i + \frac{1}{2}) \theta = \frac{b}{500} \sum_{i=0}^{\infty} \tilde{D}_i \rho^{i+1/2} \cos(i + \frac{1}{2}) \theta, \quad (5.10)
\]

Since \(r = \sqrt{x^2 + y^2}\), we have \(r = \rho a\). Eq. (5.10) is reduced to

\[
\sum_{i=0}^{\infty} \left\{ \alpha_i - \frac{b}{500} \phi_i^{-(i+1/2)}\tilde{D}_i \right\} \rho^{i+1/2} \cos(i + \frac{1}{2}) \theta = 0. \quad (5.11)
\]

Since functions \(\rho^{i+1/2} \cos(i + 1/2) \theta\) are linearly independent, we obtain

\[
\alpha_i = \frac{b}{500} \phi_i^{-(i+1/2)}\tilde{D}_i = 0, \quad (5.12)
\]

which is the desired equation (5.5).

By means of Eq. (5.5), the coefficients \(\alpha_i\) can be obtained for \(a = 1/2\) and \(b = 0.125\), which are listed in Table 13. Our leading coefficients

\[
\alpha_0 = 0.19111863197187209, \quad (5.13)
\]

\[
\alpha_1 = -0.1181160719665095, \quad (5.14)
\]

from Table 13 have 17 and 16 significant digits, respectively, compared with the more accurate values:

\[
\alpha_0 = 0.19111863197187208906830, \quad (5.15)
\]

\[
\alpha_1 = -0.11811607196650946846348.
\]

Eq. (5.15) possessing 23 significant digits are cited from Ref. [22] by the same collocation Trefftz method but using higher working digits under Mathematica. Besides, the significant digits of other coefficients are also provided in Table 13, compared with more accurate \(\alpha_0\) in Ref. [22].

We have also completed the direct computation for the traditional cracked beam problem in Fig. 3. The errors, condition numbers and the leading coefficients are listed in Tables 14–16. Interestingly, in Table 15, the same \(\alpha_0\) and \(\alpha_1\) as Eq. (5.13) are obtained. Let us compare two approaches: (1) from Table 12 by Eq. (5.5), (2) direct computation from Fig. 3. The global errors from Table 11 are 10 times smaller than those from direct computation (Table 16). On the other hand, the leading coefficients \(\alpha_0\) and \(\alpha_1\) from direct computation are slightly better than those in Table 13. We note that the condition number from the direct computation is huge, and the ratio of condition numbers between these two approaches is

\[
\frac{\text{Cond.}_{\text{Direct}}}{\text{Cond.}_{\text{From Table 11}}} = 0.186(10) / 0.447(6) = 416.
\]

Hence, the approach from Table 12 seems to be superior.

In Ref. [5], the integrated singular basis method (ISBFM) and the integral method are used to seek the solutions of the traditional cracked beam problem, and their leading coefficients are listed in Table 17 with the number of significant digits. Evidently, the leading coefficients in Tables 13 and 15 have more significant digits than those in Table 17.

In this paper, the same \(S\) is chosen for both Motz’s and the cracked beam problems, in order to unify the theoretical framework and to do comparisons. Let us look at the coefficients in Tables 5 and 12. The coefficients \(D_i\) decrease monotonically in magnitude as \(i \to \infty\), but \(\tilde{D}_e\) do not. However, each of \(\tilde{D}_e\) and \(\tilde{D}_{eq+1}\) does decrease monotonically. We have carefully checked the coefficients from Tables 5, 12 and Refs. [15, 22] to find the following empirical asymptotes

\[
\tilde{D}_e \leq C_0 \times 2x^{-4/1}, \quad \tilde{D}_{eq} \leq C_1 \times 2x^{-4e}, \quad \tilde{D}_{eq+1} \leq C_2 \times 2x^{-4e}, \quad (5.15)
\]
where $C_0$, $C_1$ and $C_2$ are positive constants. Hence, we may assume the true coefficients $d_i$ in Eq. (1.4) also satisfy the same asymptotes

$$d_i = C_i + e^{-i}$$

where $e$ is a small positive constant. Rewrite Eq. (1.4) as the sum of

$$\sum_{i=0}^{N} d_i r^{i+1/2} \cos (i + \frac{1}{2}) \theta,$$

and the remainder

$$R_N = \sum_{i=N+1}^{\infty} d_i r^{i+1/2} \cos (i + \frac{1}{2}) \theta,$$

Below, we show the following exponential convergence

$$\|u - \tilde{u}_N\|_{1,5} = \|R_N\|_{1,5} \leq C_3 \left( \frac{\sqrt{2}}{2} \right)^N,$$

where $C_3$ is a constant independent of $N$.

Denote a half-disk domain $S_{\pi} = \{(r, \theta) | 0 \leq r \leq R, 0 \leq \theta \leq \pi\}$. Then $S \subset S_{\pi}$. We have

$$\|R_N\|_{1,5} \leq \|R_N\|_{1,5} \leq C_3 \left( \frac{\sqrt{2}}{2} \right)^N,$$

By using the orthogonality of trigonometric functions, we obtain

$$I_1 = \int_0^\pi \int_0^R \left( \frac{\partial R_N}{\partial r} \right)^2 r \ dr \ d\theta$$

$$= \frac{\pi}{4} R \sum_{i=N+1}^{\infty} \left( i + \frac{1}{2} \right) R^{2i} d_i^2,$$

Then, we have

$$\|R_N\|_{1,5} \leq (I_1 + I_2 + I_3)_{R=0}$$

$$\leq \sqrt{2} \pi \sum_{i=N+1}^{\infty} \left( i + \frac{1}{2} \right) \sqrt{2} d_i^2.$$

Under Eq. (5.16), we have

$$\|R_N\|_{1,5} \leq C \sum_{i=N+1}^{\infty} \left( i + \frac{1}{2} \right) \left( \frac{2}{1 + e} \right)^{2i}.$$
For the cracked beam in Fig. 2, the same exponential convergence rates hold as Eqs. (5.19), (5.30) and (5.31). The numerical rates $O(0.56^N)$ in Eqs. (4.8) and (4.9) are coincident with the a posteriori estimates $O(0.707^N)$. For the traditional scaled cracked beam in Fig. 3, the coefficients in Table 13 have

$$\alpha_i \leq C \left( \frac{2}{3 + \epsilon} \right)^i.$$  \hspace{1cm} (5.34)

By noting that $\rho \leq \sqrt{2}$, the same exponential convergence rates as Eqs. (5.19), (5.30) and (5.31) can also be obtained.

For the Laplace equations on sectors of disks, half-disks and disks, the exponential convergence rates of the expansion solutions are proven theoretically in Ref. [23, p. 41]. The proof for the exponential rates on the rectangular domains in $S$ and $\hat{S}$ is given in this section by the a posteriori analysis, where the assumption (5.16) is purely based on numerical observation of the obtained results. For the rigorous proof of exponential convergence rates on $S$ without Eq. (5.16) needs to be further explored.

6. Concluding remarks

To close this paper, let us make a few remarks.

1. Computational algorithms of the collocation Trefftz method are provided in Section 2. The overdetermined system (2.28) is recommended in computation since its algorithm is simple and easy, which is, indeed, just the collocation method at the boundary nodes, based on Proposition 2.1. The remarkable advantage of Eq. (2.28) is that the condition numbers of the associated matrix can be dramatically reduced, compared to Eq. (2.37) of the normal equation.

2. Different quadratures, such as the central and Gaussian rules, are investigated for the LSM. Theorem 3.1 reveals that different integration rules do not make much differences in the global errors over the entire domain $S$. However, the rules used may affect significantly the accuracy of the leading coefficient, based on numerical experiments in this paper.

3. The quadrature is used to link the collocation method and the LSM. However, from our error analysis, the accuracy of a quadrature may be very rough, in the sense that its relative errors are less than three quarters! This feature is significantly different from the traditional integral approximation in error analysis, e.g. the FEM analysis, where the integration errors should be chosen to balance the optimal errors of the solutions. Based on the analysis in Section 3, the solutions of Motz’s and the cracked beam problems solved by the collocation Trefftz method have the exponential convergence rates. Note that Theorem 3.1 and Proposition 3.1 are new, which provide a theoretical foundation for high accuracy of the collocation Trefftz method (e.g. the BAM). This is also a justification for the collocation Trefftz method to become the most accurate method for Motz’s and the cracked beam problems. Besides, the collocation methods both in $S$ and on $\partial S$ are explored in Ref. [8].

4. The numerical results in Section 2 are better than those in Refs. [14,16]. The Gaussian rule with six nodes are used to raise the accuracy of the leading coefficient to

$$D_0 = 401.162453745234416$$  \hspace{1cm} (6.1)

by the collocation Trefftz method. Compared with the more accurate value of $D_0$ in Refs. [14,15], this $D_0$ has exactly 17 significant digits, which error happens to coincide with the rounding errors of double precision. Note that coefficient $D_0$ in Ref. [16] has only 12 significant digits. This new discovery will change the evaluation of the BAM (i.e. the collocation Trefftz method) given in Ref. [14]. Based on the numerical results in Ref. [16] using the central rule, it is pointed out in Ref. [14, p. 133] that “BAM may produce the best global solutions”, but “the conformal transformation method is the highly accurate method for leading coefficients”. Now we may address that for Motz’s problem, the collocation Trefftz method (i.e. the BAM) by the Gaussian rule of high order is a highly accurate method, not only for the global solutions but also for the leading coefficient $D_0$.

5. The new numerical results by the collocation Trefftz method in Section 4 provide a highly accurate solution for the cracked beam problem. The Gaussian rules of high order are used to raise the accuracy of the leading coefficient to

$$\hat{D}_0 = 540.56512271362749,$$  \hspace{1cm} (6.2)

which also has 17 significant digits. For the traditional cracked beam problems in Fig. 3, coefficients

$$\alpha_0 = 0.19111863197187209$$  \hspace{1cm} and  \hspace{1cm} $\alpha_1 = -0.1181160719665095$  \hspace{1cm} (6.3)

from Table 13 have 17 and 16 significant digits, respectively. Thus the collocation Trefftz method using the Gaussian rule of high order is also a highly accurate method for the cracked beam problem, not only for the global solutions but also for the leading coefficient $\hat{D}_0$.

6. Motz’s and the cracked beam problems are linked and compared by considering the same domain $S$. The traditional crack model in Refs. [4–6,19,21] is formulated as a special case of the scaled cracked beam problem in this paper, whose solutions can be obtained straightforward by Eq. (5.5). Besides, numerical experiments from the approaches by Table 12 using Eq. (5.5) and by direct computation for Fig. 3 have been reported. The former seems to be superior due to its smaller condition numbers. Motz’s and the cracked beam problems are regarded as the Laplace equations on $S$ with two different boundary conditions along the edges.
Hence, different boundary conditions on $\partial S$ may have different impacts on the singular behavior of the Laplace solutions in $S$. These computational results will appear later.

7. There was a special issue on the Trefftz methods, i.e. Advanced in Engineering Software, vol. 24, 1995. Some overviews can be found in Refs. [9,11,24]. In Refs. [9,11], the Trefftz methods are classified into the indirect and direct methods. The collocation Trefftz in this paper is just the indirect Trefftz method. We use the same terminology, the Trefftz collocation method, as in Ref. [13]. The direct Trefftz method is analogous to the boundary element method except the fundamental functions are replaced by the singular function in the trial space. We report in this paper the new computational results and the new analysis of the indirect Trefftz method. In some extent, we have filled up the gap existing before between excellent computation and theoretical analysis of this method [10].

Acknowledgements

We are grateful to Prof. Alexander H.-D. Cheng for valuable communication, and also appreciate the reviewers for their very valuable comments and suggestions.

References