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Abstract—We report the instability behaviors of a single-mode microchip solid-state laser subjected to external feedback. Two kinds of instabilities, random chaotic burst generations and random sinusoidal burst generations, were observed experimentally in an LD-pumped microchip Nd:YVO₄ single-mode solid-state laser with fiber feedback. These results are totally different from those observed in laser diodes with delay feedback systems, which have been widely studied in last decades. Main features were reproduced numerically by utilizing the Lang–Kobayashi equations with phase noise, indicating phase-noise-driven dynamic instabilities.

Index Terms—Laser stability, neodymium, numerical analysis, optical feedback, phase noise, solid lasers, stochastic differential equations.

I. INTRODUCTION

In history, the issue of instabilities in the output of lasers that are subjected to external feedback was initiated by the pioneering work of Lang and Kobayashi in 1980 [1]. They demonstrated the dynamic instabilities in a laser diode with external feedback which feature sustaining relaxation oscillations (ROs). They also confirmed theoretically that the dynamical instabilities take place in the transition process where the lasing frequency changes from one external cavity eigenmode to another in a weak-coupling regime. Thereafter, three universal transition routes to chaos, low-frequency fluctuations, and coherence collapse have been observed in LDs with external optical feedback for different feedback strength and/or delay time [2].

Some recent studies on the influence of optical feedback on LDs have focused on reducing or controlling the destabilizing effects of feedback [3]–[7]. Very weak feedback from short external cavities can significantly reduce both intensity noise and lasing linewidth [8]. The emission frequency can also be stabilized with phase-conjugate feedback [9]. On the other hand, variation of the feedback within the range that leads to chaotic operation can be used to encrypt information [10], [11]. Furthermore, feedback into a broad-area laser can bias the selection of a particular lateral mode, though this is sensitive to very small variations of the external-cavity length [12], [13]. Meanwhile, pulse-to-pulse jitter in spontaneously pulsing or externally switched LDs can be reduced with optical feedback, but the reduction is also extremely sensitive to small variations in cavity length [14], [15]. In contrast, studies on LD-pumped solid-state lasers with external optical feedback are still scant. Because LD-pumped solid-state lasers have been widely used in practical applications, it is important to clarify their dynamical behaviors both for academic and application reasons. Fiber linking capability of microchip lasers could be an important issue for practical applications. Thus, a characterization of the dynamics behind fiber-feedback-induced instability is crucial.

The time scales of the intensity fluctuations of LDs are much less convenient for precise dynamical measurements than those of solid-state lasers. Because the characteristic frequencies of solid-state lasers are sub-MHz, they are much more convenient for measurement. They also exhibit extremely high-sensitive responses to external feedback. The reason is that the cavity round-trip time \( \tau_L \propto \tau_S \) (photon lifetime) compared with the fluorescent lifetimes \( \tau_f \) is extremely short [16]. Generally, the lifetime ratio \( K = \tau_f/\tau_S \) of solid-state lasers ranges from \( 10^3 \) to \( 10^5 \), while \( K \sim 10^3 \) in LDs. Hence solid-state lasers become another promising laser systems for investigating the instabilities in lasers with delay feedback. Because the time scales of solid state lasers are dramatically different from those of LDs, the dynamical behaviors are expected to be unique. In fact, in the early experiment in which a microchip LNP (LiNdP₃O₁₂) solid-state laser was subjected to external feedback [17], feedback-induced random chaotic burst generations were observed. However, the instability is only observed in the regime of two lasing modes. Most recently, the random chaotic burst state has been re-examined using an LD-pumped LNP laser coupled to a single-mode optical fiber in a weakly-coupled regime, in which only one of the external cavity modes interacts with the solitary mode. It is conjectured that mode-partition noise in multimode regimes plays an essential role for producing the random chaotic burst state since phase fluctuation (FM) noise is extremely small in single-mode LNP lasers [18].

The results of the solid-state laser experiments cannot be interpreted in the context of low-frequency fluctuations in LDs described by the single-mode Lang–Kobayashi equations. Low-frequency fluctuations occur for moderate feedback when the laser is biased near the solitary laser threshold \( (I/I_{th} < 1.2 \sim 1.4) \). In this regime, successively the laser intensity suddenly drops toward zero and gradually recovers with apparently random time length. The physical mechanism
behind low-frequency fluctuations is still not fully understood, though several different explanations have been proposed. Mørk, Tromborg, and Christiansen explained low-frequency fluctuations as the result of bistability among the steady-state solutions introduced by the external cavity, i.e., external cavity modes [19]. Hohl, van der Linden, and Roy showed that spontaneous emission noise plays an important role in the nature and the statistics of the dropouts [20]. The experimental measurements agreed with those of Henry and Kazarnin, suggesting that spontaneous emission noise induces the dropout events [21]. On the other hand, Sano explained the dropouts as a result of a self-induced switching among distinct regions of phase space [22]. In Sano’s interpretation, the laser moves toward the mode with maximum gain via chaotic itinerancy. However, since in the neighborhood of the maximum gain mode (stable external cavity mode) and antimodes (i.e., unstable external cavity modes) are very close, when the trajectory approaches an antimode, it is expelled into another region of phase space, and then starts moving toward the maximum gain mode again [23], [24]. Recent high-speed measurements on low-frequency fluctuations indicated that “power dropout” are seen as the envelope of a series of short pulse with 100 ps or less pulsewidth [24]. Similar pulses have been found in the Lang–Kobayashi model [23], [24]. In the meantime, temporally resolved optical spectra reveal that there is enhanced power in several longitudinal modes during the power dropout [25].

However, as will be shown below, the dynamical features of the instability of LD-pumped solid-state lasers with optical feedback are totally different from the instabilities (low-frequency fluctuations) in LDs. The instabilities reported below occur even in high pumping levels rather than near threshold only. On the other hand, instability occurs even with very weak feedback strength in which basically only one of the external cavity modes interacts with the solitary laser mode. More precisely, a variable attenuator was inserted.

In this paper, we report two kinds of random switching in an LD-pumped microchip neodymium-doped yttrium orthovanadate (Nd:YVO₄) single-mode laser coupled to a single-mode fiber. Two kinds of instabilities were observed in different feedback strength regimes, though the threshold reduction caused by feedback is less than 1%. In the weaker feedback regime, the laser output randomly switches between noise-driven relaxation oscillation and chaotic spike oscillation operations. In the stronger feedback regime (but the threshold reduction is still less than 1%), the laser output randomly switches between noise-driven relaxation oscillation and large-amplitude sinusoidal oscillation operations. The distinct features which are different from previous results are that the random burst generations were observed in a single-mode regime while in LNP lasers they were restricted to the multi-mode regime [17], [18] and the random chaotic burst occurred independently of the pump power level in weakly-coupled regimes in which only a single external cavity mode exists while low-frequency fluctuations in LDs has been observed only in the vicinity of solitary laser threshold in which many external cavity modes were involved in the dynamics. This paper is organized as follows. In Section II, the experimental scheme is described. The coupling coefficient is provided based on an estimate according to our experiment setup and main characteristics of the solitary Nd:YVO₄ laser are first summarized. The experimental results are presented in Section III. The random chaotic burst and the random sinusoidal burst are reported and their characteristic features are discussed. To explore experimental results, we utilize the Lang–Kobayashi equations including strong phase noise in numerical simulation and the observed random chaotic burst instability can be well reproduced as shown in Section IV. It is worthwhile to note that we employed time scales involving dynamics in microchip solid-state lasers, i.e., fluorescent lifetime, photon lifetime, and delay time, which are totally different from LDs. Section V describes the conclusions of our work.

II. EXPERIMENTAL SETUP

The experimental setup is shown in Fig. 1. An LD-pumped microchip Nd:YVO₄ laser operated in the single-mode regime was employed and a compound cavity was formed with a single-mode fiber. The laser crystal (CASIX DPO3104; 1 mm thick, 1% Nd³⁺ doped, and 5 ± 2% output coupling at 1064 nm) was inserted into a 2-mm-thick copper mount and its temperature was controlled at 25°C by a temperature controller. The pump beam of λp = 808 nm from the temperature-controlled LD was focused onto the laser crystal with a GRIN lens (0.22 pitch). The pumping threshold was around 40 mW and single-mode operation ranged up to 100 mW. Above 100 mW, the laser operates in a multi-mode regime (two modes and three modes were observed for different pumping powers). We also used a noise filter to eliminate the pumping noise caused by the LD controller and an interference filter to reduce the influence of pumping light on detection. In the entire pumping domain a linear π-polarized TEM₀₀ mode of laser output was observed. The mode suppression ratio of the polarization mode is about 1/300. The laser beam was divided into two by a 50%-50% beamsplitter plate following after the interference filter. One was for measurement and the other was for feedback. We used a single-mode fiber (operating wavelength 1060 nm, core/cladding diameter 6.4/125 μm, cutoff wavelength 970±60 μm, maximum attenuation 2.0 dB/km) as the feedback reflector such that the laser light reflected back from the far end of the fiber, in which the maximum field reflectivity is about 0.2. The coupling efficiency of the laser light into the fiber is estimated to be about 50%. To control the feedback strength more precisely, a variable attenuator was inserted.

The feedback strength can be estimated by threshold reduction. But to further quantify weak feedback, threshold reduction is not significant. Thus, we use alternative to handle the issue. The coupling coefficient κ can be estimated with the formula [26]

\[
κ = \frac{r_{\text{ext}}}{\tau_L r_o} (1 - |r_o|^2),
\]

where

- \(\tau_L\) laser cavity round-trip time and is about \(1.33 \times 10^{-11}\) s for a 1-mm-thick Nd:YVO₄ chip (the index of refraction: \(n_o = 1.9573\));
- \(r_o\) field reflectivity of the laser output mirror (about 0.975 for high-reflection coated);
- \(r_{\text{ext}}\) total field reflectivity of external components.
Referring to the experimental setup, $r_{\text{ext}}$ can be estimated as $r_{\text{ext}} = r_{b} \cdot t_{a} \cdot n_{c} \cdot r_{f-a} \cdot \eta_{c}$. Here, $r_{b}$ is the field reflectivity of the beam splitter and is about 0.7 for a 50%–50% beam splitter. $t_{a}$ is the transmission coefficient of the variable attenuator and the value is variable from 0 to 0.95. $n_{c}$ is the coupling efficiency into the fiber. $r_{f-a}$ is the field reflectivity of the fiber–air interface and the value is about 0.2. Hence, the coupling coefficient $\kappa$ is smaller than $1.7 \times 10^{6}$ s⁻¹. We will use this parameter $\kappa$ to quantify our experiment.

In measurement, a multi-wavelength meter (resolution of 20 GHz) was employed to monitor the variation of the lasing mode. The single solitary mode operation was confirmed. The lasing wavelength $\lambda_{l}$ was 1064.245 nm. In the following experiment, single mode operation was maintained. A high-resolution scanning Fabry–Perot interferometer (resolution of 0.185 MHz) was used to identify the detailed structure of the oscillation spectrum of the laser output. The Nd:YVO₄ laser always emits a linearly polarized light due to the strong fluorescence anisotropy. The beamsplitter plate we used did not affect the polarization of laser light. The linear polarization was confirmed to be maintained after passing through the optical fiber because the fiber length was 10 m at most. Because the system was very sensitive to optical feedback, a Faraday isolator (isolation of 60 dB) was employed to isolate the feedback light from the measurement part. We utilized a low-noise photodetector (typical rise time: < 1 ns) for detection and connected it to a transient oscilloscope (bandwidth: 500 MHz) for data acquisition. The GPIB interface was implemented to catch data from the oscilloscope with PC. An RF spectrum analyzer (bandwidth: 9 kHz–1.8 GHz) was used to monitor the behavior of laser output in the RF spectrum domain.

For comparison, typical output characteristics of the solitary Nd:YVO₄ laser are shown in Fig. 2, where the pumping power was 88 mW. In the absence of feedback, as shown in Fig. 2(a), the laser output exhibits noise-driven relaxation oscillation. The
corresponding power spectrum is shown in Fig. 2(b). The RO frequency, $f_{\text{RO}}$, is around 1.7 MHz and its second harmonic is at about 3.4 MHz. The oscillation characteristics were identified by utilizing the high-resolution scanning Fabry–Perot interferometer. In Fig. 2(c), the corresponding oscillation spectrum is presented and the full-width at half-maximum (FWHM) linewidth of the laser is estimated to be 2.2 MHz. Hereafter, the linewidth is denoted based on FWHM. As the pumping power was increased, the relaxation oscillation frequency increased following the relation, $f_{\text{RO}} = \frac{2\pi f_{\text{RO}} \times \sqrt{P/P_{\text{th}} - 1}}{\tau_s \tau_f}$, where $P$ and $P_{\text{th}}$ denote the pumping power and the threshold pumping power, respectively.

Several of the other optical properties of the Nd:YVO$_4$ crystal we used in our experiments are: diode-pumped optical-to-optical efficiency was $> 60\%$; fluorescent lifetime $\tau_f = 90\ \mu$s; absorption coefficient $= 31.4\ \text{cm}^{-1}$; absorption length $= 0.32$ mm; gain bandwidth $= 0.96$ nm (257 GHz); intrinsic loss $= 0.02\ \text{cm}^{-1}$; stimulated emission cross-section $= 2.5 \times 10^{-20}\ \text{cm}^2$.

### III. General Features of the Experimental Results

#### A. Weaker Feedback Strength ($\kappa < 2.5 \times 10^7\ \text{s}^{-1}$)

We fixed the pumping power at 88 mW in all our experiments. In the presence of fiber feedback and when $\kappa < 1.1 \times 10^5\ \text{s}^{-1}$, the dynamic features are almost the same as those of the solitary laser. When $\kappa > 1.1 \times 10^5\ \text{s}^{-1}$, random chaotic burst occurs. A typical temporal waveform of random chaotic burst is shown in Fig. 3 for $\kappa = 5.8 \times 10^5\ \text{s}^{-1}$ with a 5-m fiber. The laser output randomly switches between noise-driven relaxation oscillation and chaotic spike oscillation. The duration time of chaotic spike oscillation is also irregular. The insert shows the short temporal trace of chaotic spike oscillation. Here, the irregular spiking oscillation waveform can be seen and has been identified to be chaos based on singular value decomposition analysis [27]. The power spectrum of noise-driven relaxation oscillation is shown in Fig. 4(a). This figure exhibits a typical noise-driven relaxation oscillation power spectrum with $f_{\text{RO}}$ being also around 1.7 MHz. Fig. 4(b) shows the power spectrum of chaotic spike oscillation. One can see that the main frequency component shifts to a lower value (around 1.2 MHz) and the spectrum broadens. In the transition region, which shows the evolution from noise-driven relaxation oscillation to chaotic spike oscillation, the power spectrum broadens and the main frequency component shifts from $f_{\text{RO}}$ to a lower frequency value which partially reflects the nature of chaotic spike oscillation.

If we increase the feedback strength further, the $f_{\text{RO}}$ of noise-driven relaxation oscillation, together with the dominant frequency of the chaotic spike oscillation, will shift to lower frequency values. Fig. 5 shows the power spectra of chaotic spike oscillation with different feedback strengths, $\kappa = 1.1 \times 10^6\ \text{s}^{-1}$,
Fig. 6. Temporal waveform of random sinusoidal burst ($\kappa = 5.5 \times 10^7$ s$^{-1}$). The insert shows the short temporal trace of large-amplitude sinusoidal oscillation.

$5.2 \times 10^6$ s$^{-1}$, and $6.8 \times 10^6$ s$^{-1}$. The main frequency component shifts to lower values slightly due to the increase of the feedback strength as indicated by the line crossing the figures. The range of the frequency shift in this regime is about 0.2 MHz. This feature differs from the previous results in LDs that $f_{FB}$ increases with an increase of feedback strength [28]. As will be discussed below, this is due to the time scale difference between LDs and solid-state lasers.

By utilizing a high-resolution scanning Fabry–Perot interferometer with suitable triggering, we can characterize the linewidth of the laser output in different regions. As shown by Fig. 4(c), the linewidth corresponding to noise-driven relaxation oscillation in Fig. 3 is about 2 MHz which is close to that of the solitary laser. On the other hand, Fig. 4(d) shows the corresponding oscillation spectrum of chaotic spike oscillation of Fig. 3, and the linewidth broadens to about 5 MHz. Because the lasing linewidth is mainly due to the phase fluctuation, this implies that the phase fluctuation becomes larger as chaotic spike oscillation occurs.

B. Stronger Feedback Strength ($2.5 \times 10^7 < \kappa < 1.7 \times 10^8$ s$^{-1}$)

For stronger feedback strength, i.e., $\kappa > 2.5 \times 10^7$ s$^{-1}$, another kind of burst was observed and the burst mentioned in Section III–A disappeared. For $\kappa \sim 2.5 \times 10^7$ s$^{-1}$, these two different bursts can coexist. In Fig. 6, the output waveform observed at $\kappa = 5.5 \times 10^7$ s$^{-1}$ is shown. Here, random sinusoidal burst is observed. The laser output randomly switches between noise-driven relaxation oscillation and large-amplitude sinusoidal oscillation, and the duration time of large-amplitude sinusoidal oscillation is also irregular. The insert shows the short temporal trace of large-amplitude sinusoidal oscillation. The large-amplitude oscillation exhibits a sinusoidal waveform. The corresponding power spectra of Fig. 6 are shown in Fig. 7. Fig. 7(a) shows the power spectrum of noise-driven relaxation oscillation. $f_{FB}$ is shifted by $\sim 0.9$ MHz to a lower frequency value due to stronger optical feedback. In contrast, as shown in Fig. 7(b), which is corresponds to large-amplitude sinusoidal oscillation, a sharp peak which denotes the frequency of the sinusoidal waveform is found at about 12.5 MHz. Its value increases with an increase of the feedback strength and the range of the frequency shifting will be discussed latter.

Next, we report the oscillation spectra in this regime. In contrast to weaker feedback strength (i.e., $\kappa < 2.5 \times 10^7$ s$^{-1}$), in which only linewidth broadening was observed, a random switching between single-mode and two-mode operations was observed. Figs. 7(c) and (d) show the corresponding oscillation spectra of Fig. 6. Fig. 7(c) shows the single-mode operation with a linewidth of 2.1 MHz, which approximates that of the solitary laser. Fig. 7(d) shows the case of two-mode operation. The oscillation frequency difference between two modes is about 12.8 MHz, as indicated by the down arrows. This value also approximates the frequency of the sinusoidal waveform. In this regime, the frequency of the sinusoidal waveform and the oscillation frequency difference between two observed modes are approximately equal and vary with the feedback strength.

To explore the influence of delay time (fiber length) on the experimental results, we further used two additional fibers with different lengths of 10 and 2 m. In the weaker feedback regime, $\kappa < 2.5 \times 10^7$ s$^{-1}$, the dynamical features are almost the same as those of the 5-m fiber. However, in the stronger feedback regime, $\kappa > 2.5 \times 10^7$ s$^{-1}$, as the feedback strength was fixed, both the frequency of the sinusoidal waveform and the oscillation frequency difference between the two observed modes varied with the delay time. Fig. 8 shows power spectra and oscillation spectra of the laser output for different delay times, where $\kappa = 5.5 \times 10^7$ s$^{-1}$. Fig. 8(a) is the power spectrum for a 10-m fiber with a delay time $10^{-7}$ s and Fig. 8(c) is the corresponding oscillation spectrum. Both the frequency of sinusoidal waveform and the oscillation frequency difference between two modes are 5.8 MHz. Fig. 8(b) shows the power spectrum for a 2-m fiber (corresponding delay time: $2 \times 10^{-5}$ s) and Fig. 8(d) is the oscillation spectrum, while both the frequency of sinusoidal waveform and the oscillation frequency difference
between two modes are 20.3 MHz. The frequency of the sinusoidal waveform coincides with the oscillation frequency difference between two observed modes and they vary inversely with the delay time when the coupling coefficient is fixed. Because the scanning rate of the scanning Fabry–Perot is 30 Hz, the oscillation spectra were measured in an average of 1/30 s; how these two modes interact with each other could not be specified by the measurements. However, these measurements strongly suggest that these two observed modes are nothing other than external cavity modes [which can be derived from (5)] of the compound cavity to be discussed in the next section. Note that the mode spacing does not coincide with the simple free-spectral range of the external cavity. Furthermore, the nonlinear interaction between the two stable external cavity modes is considered to result in large-amplitude sinusoidal oscillations.

Although the results reported above were for a pumping power at 88 mW, exactly the same dynamics were observed independently of the pumping power levels in a single-mode regime. In short, in LD-pumped Nd:YVO$_4$ microchip single-mode laser with fiber feedback, two different kinds of instability, i.e., random switching between noise-driven relaxation oscillation and chaotic spike oscillation (random chaotic burst state) and random switching between noise-driven relaxation oscillation and large-amplitude sinusoidal oscillation (random sinusoidal burst state) were observed for different feedback strengths. In the next section, we will explore the observed dynamics based on the Lang–Kobayashi model.

### IV. THEORETICAL EXPLORATION BASED ON THE LANG–KOBAYASHI MODEL

#### A. General Characteristics of the Lang–Kobayashi Model

Since our system is essentially a single-mode laser with weak feedback, we employed the Lang–Kobayashi model [1] to explore the dynamics. We used the normalized Lang–Kobayashi equations with Langevin noises [20], [29], [30]

\[
\frac{dS(t)}{dt} = \kappa \left[ (N(t) - 1)S(t) \right] + 2\kappa \sqrt{S(t)S(t-T)} \cos(\Theta(t)) + F_S(t) \\
\frac{dN(t)}{dt} = W - N(t) - N(t)S(t) + F_N(t) \\
\frac{d\phi(t)}{dt} = \Delta \omega + \frac{1}{\alpha}K[N(t) - 1] \\
- \kappa \sqrt{\frac{S(t-T)}{S(t)}} \sin(\Theta(t)) + F_\phi(t)
\]

where \(\Theta(t) = \omega T + \phi(t) - \phi(t-T)\) and \(\omega = \omega_\text{th} + \Delta \omega\) is the optical angle frequency of the solitary laser, \(\omega_\text{th}\) is the optical angular frequency of solitary laser near threshold, and \(\Delta \omega\) is the optical angular frequency shift of the laser deviated from the threshold. Equations (2)–(4) are the temporal evolution equations for the photon density \(S\), the slowly varying part of the optical phase \(\phi\), and the population inversion density \(N\). In the rate equations, \(t\) has been normalized to \(\tau_F\), \(W = P/\bar{P}_\text{th}\) is the normalized pumping rate, \(\kappa\) is the time ratio between \(\tau_F\) and \(\tau_S\), \(\alpha\) is linewidth enhancement factor, \(T\) is the delay time, and \(\Theta\) is the phase difference between the output and the feedback beams [33]. \(F_S\), \(F_N\), and \(F_\phi\) are the Langevin noise sources which satisfy \(\langle F_i(t) \rangle = 0\) and \(\langle F_i(t)F_j(t') \rangle = 2D_{ij}\delta(t-t')\). The angle brackets denote a time average and \(D_{ij}\) is the diffusion coefficient associated with the corresponding noise source. Here, the subscripts \(i,j = S, N, \phi\). Their explicit expressions are

\[
D_{SS} = \varepsilon S, \quad D_{\phi\phi} = \frac{\varepsilon}{4S}, \quad D_{S\phi} = 0 \quad D_{NN} = \varepsilon S + N, \quad D_{SN} = -\varepsilon S, \quad D_{NN} = 0
\]

where \(\varepsilon\) is the spontaneous emission coefficient. The stationary solutions of (2)–(4) are external cavity modes. The stationary optical angular frequency is the solution of

\[
\Delta \omega T + C \sin(\omega T + tan^{-1}^\alpha) = 0
\]

where \(C \equiv \kappa T \sqrt{1+\alpha^2}\) is called effective feedback strength. The linewidth enhancement factor \(\alpha\) in LDs stems from the combined effect of free-carrier plasma effect and detuning effect of lasing frequency from the spontaneous emission peak, i.e., the Kramers–Kronig relationship. In microchip solid-state lasers, no one has determined its value experimentally. However, recent experimental demonstration of large phase-conjugate reflection from Nd:YVO$_4$ lasers operating near gain peak [32] indicates that the Nd:YVO$_4$ laser itself has a substantially large \(\chi^3\) nonlinearity which is directly related to a \(\alpha^2\)-nonlinearity in the form of \(\chi^3 = (\alpha - j)(G_0/I_s)\) where \(G_0\) small-signal gain, \(I_s\) saturation intensity\( j = \sqrt{-1}\). Therefore, we will use a nonzero \(\alpha\) in the simulation. The steady-state characteristics can be characterized by \(C\). From
Fig. 9. Typical numerical characteristics of $S(t)$ based on the Lang–Kobayashi model with $\kappa = 0$. (a) Temporal waveform. (b) Power spectrum. (c) Oscillation frequency histogram.

As (6) does not hold, the solution is intrinsically unstable (i.e., a saddle point) and is called antimode [22]. However, it should be noted that external cavity modes are not always dynamically stable even if this quantity is positive [33] and their stabilities change depending on $\Delta \omega T$ in (5). In short, dynamical stability of an external cavity mode depends critically on the solitary laser frequency $\omega_0$ in a large delay case as in our present experiment.

The linewidth in the presence of external feedback is approximately given by [34]

$$\Delta \nu_f \approx \frac{\Delta \nu}{1 + C \cos(\omega T + \tan^{-1}\alpha)}$$

where $\Delta \nu$ is the solitary laser linewidth. Equation (7) shows that, depending upon the feedback phase $\omega T$, the laser linewidth can broaden or narrow. The maximum narrowing occurs when $\omega T + \tan^{-1}\alpha = 2m\pi$, where $m$ is an integer, and the reduction factor in that case is $(1 + C)^2$. Clearly the linewidth can be reduced considerably for a large value of $C$ if the external cavity length is fine-tuned to adjust $\omega T$. Reductions by more than three orders of magnitude have been observed [35]. As discussed earlier, when $C > 1$, there are multiple external cavity modes over which the laser can be made to oscillate by changing the external cavity length [36]. In the case of mode hopping, the linewidth can vary by a large amount with little change in $T$ [37]. For $C < 1$, only one external cavity mode oscillates for all values of the phase. Considerable linewidth broadening can occur in this case for certain range of phase values, as seen clearly by an inspection of (7). However, (7) is very sensitive to the feedback phase $\omega T$. For the parameters of our experimental condition, i.e., $\lambda_0 = 1064.245$ nm, $\omega T \approx (1.18 \times 10^{10} \times L)$ rad, where $L$ is the effective external cavity length. Even a 1-µm variation of the effective external cavity length can vary $\omega T$ for 7.7 rad. We did not fine-tune this value in the experiments. Instead, we just paid attention to the change of the lasing linewidth as instabilities occurred. Nevertheless, we did not observe any significant linewidth narrowing. On the other hand, linewidth broadening with a maximum $(\Delta \nu_f) / \Delta \nu \approx 2.5$ occurred only when chaotic spike oscillation appeared.

The chief difference between solid-state lasers and LDs are the time scales. The relevant time scales in our laser system are: $\tau_f \approx 90 \times 10^{-6}$ s, $\tau_s \approx 2 \times 10^{-20}$ s, $f_{HO} \approx 1.7$ MHz, decay time $\approx 876 \times 10^{-3}$ s, the internal round-trip time $= 1.33 \times 10^{-11}$ s for a 1-mm-thick crystal, and the external round-trip time $= 3.3 \times 10^{-8}$ s for a 5-m fiber. According to these parameters, it is worthwhile to note that there are two significant differences between solid-state lasers and LDs. First, the lifetime ratio $K$ for a solid-state laser is about $10^3 - 10^4$ and $10^3$ for LDs. Second, in solid-state laser, the round-trip time is usually larger than the reciprocal of $f_{HO}$ whereas in LDs, in most cases, it is smaller than the reciprocal of $f_{HO}$. It is conjectured that these two differences are the primary reasons that result in the different behaviors between solid-state lasers and LDs with feedback.

We solved (2)–(4) numerically by the Heun method [38]. The parameters used here are: $W = 3\tau_f^{-1}$, $K = 10^6 \tau_f^{-1}$, $\alpha = 2$, $\omega_0 = 1.6 \times 10^{11}$ rad $\tau_f^{-1}$. $\Delta \omega = 10 \tau_f^{-1}$, $D_{SS} = 5 \times 10^{-9} \tau_f^{-2}$, $D_{NN} = 5 \times 10^{-5} \tau_f^{-2}$, $D_{PP} = 2 \times 10^4 \tau_f^{-2}$ and $T = 0.001 \tau_f$. The corresponding fiber length is about 9 m. To explain the experimental results, such as feedback-induced instabilities and linewidth broadening, we shall introduce a relatively large phase (i.e., FM) noise. The large phase noise derives from the spontaneous emission. Nd:YVO$_4$ lasers contain larger spontaneous emission noise so that they exhibit larger linewidth than other solid state lasers. The linewidth of the solitary Nd:YVO$_4$ laser is about 2 MHz, whereas that of...
single-mode LNP lasers is only about few hundred kilohertz. As a result, the following results could not be reproduced in a single-mode domain of LNP lasers even if a corresponding feedback strength was included. The phase noise we use corresponds to the linewidth of the Nd:YVO₄ laser.

A typical temporal waveform, power spectrum, and oscillation frequency histogram for a solitary laser, i.e., \( \kappa = 0 \), are shown in Fig. 9. Here, the oscillation frequency histogram was calculated based on (4), i.e., \( \phi(t)/\tau = \delta \omega(t) \), by using the numeric of simulated \( S(t), N(t), \) and \( \phi(t) \). \( \delta \omega(t) \) is the laser optical angular frequency fluctuation as a function of time. The probability distribution of \( \delta \omega(t) \) is plotted. The histogram may correspond to the oscillation spectrum qualitatively. The correspondence between the experiment and the simulation is fine. The numerical temporal waveform of \( S(t) \) of a solitary laser shown in Fig. 9(a) indicates noise-driven relaxation oscillation with amplitude fluctuations similar to the experimental results reported in Fig. 2(a). In the power spectrum, shown in Fig. 9(b), \( f_{BO} \) at \( 2 \times 10^7 \) and its high-order harmonic can be recognized. When we increase \( W_{V} \), \( f_{BO} \) also increases. A typical oscillation frequency histogram is shown in Fig. 9(c).

### B. Weaker Feedback Strength (\( \kappa = 10^{\tau_f^{-1}} \))

Optical feedback was introduced with \( \kappa \neq 0 \). When \( \kappa \) was small, the temporal waveform, the power spectrum, and the oscillation frequency histogram were still similar to those of a solitary laser. This parallels the experimental observation. When \( \kappa = 10^{\tau_f^{-1}} \approx 1.1 \times 10^6 \text{ s}^{-1} \) (assuming \( \tau_f = 90 \text{ ms} \)) and by tuning the \( \omega_{th} \) value precisely, the sustained relaxation oscillation spiking appeared in the absence of phase noise as shown in Fig. 10(a). Here, we assumed a negligibly small carrier density noise \( D_{N\phi} \) and a photon density noise \( D_{S\phi} \) of \( 5 \times 10^{-6} \tau_f^{-2} \). Therefore, the resulting spiking oscillations are considered to be a deterministic instability resulting from weak optical feedback. However, this spike oscillation persists, and no bursting occurs in the absence of phase noise.

When we introduced phase noise whose magnitude is relevant to that of the linewidth of the Nd:YVO₄ laser, the random chaotic burst state was reproduced quite well. An example is shown in Fig. 10(b), assuming \( D_{\phi \phi} = 2 \times 10^6 \text{ rad}^2 \tau_f^{-2} \). The output waveform switches randomly between noise-driven relaxation oscillation and chaotic spike oscillation. The corresponding power spectra and the oscillation frequency histogram are shown in Fig. 11. As Fig. 11(a) shows, the \( f_{BO} \) of noise-driven relaxation oscillation is about \( 2 \times 10^7 \) which is equal to that of the solitary laser. For chaotic spike oscillation, as shown in Fig. 11(b), the main frequency component shifts to a lower value (around \( 100 \tau_f^{-1} \)) and the broad-band characteristic is also observed. The linewidth broadened to three times as the chaotic burst occurred as shown in Fig. 11(c) and (d). This is similar to the experimental results shown in Fig. 4.

Under the present setting of parameters, \( f_{BO} \) as well as the main frequency component of chaotic spike oscillation also decreased as \( \kappa \) was increased. This result is different from that in LDs with feedback. Because the main difference between the LD and the solid-state laser are the dynamic time scales, we used another set of time scales for the LD, i.e., \( K = 10^3 \) and \( \tau_f = 10^{-9} \text{ s} \), and fixed all other parameters. For these new time scale settings, \( f_{BO} \) increases with an increase of \( \kappa \). From these results, the different trend of the dependence of \( f_{BO} \) on \( \kappa \) is due to the different dynamical time scales.

From these quantitative agreements between the experiment and the numerical results in all aspects, the chaotic bursting can be concluded to appear due to phase fluctuation of the laser (i.e., FM noise), which corresponds to fluctuations of \( \omega_{th} \). In short, random chaotic burst is thought to be random dynamic switchings between the unstable state and the stable state driven by FM noise [39].
Let us evaluate the corresponding effective feedback parameter $C$ in the present system. For the condition of numerical simulation, $\kappa = 10^7 \tau_{\text{f}}^{-1}$ and thus $C = 0.022$, which is smaller than 1. Therefore, only one external cavity mode interacts with the solitary laser mode. In our experiment, for the case of weak feedback, i.e., $\kappa = 10^6 \text{s}^{-1}$, $C = 0.071$, which is also smaller than 1. According to the context of LDs with feedback described by the single-mode Lang–Kobayashi equation, no low-frequency fluctuations instability should happen in the regime where $\tau_{\text{f}}$ is smaller than 1. Nevertheless, in the case of microchip solid-state lasers in a large delay limit, the instabilities can be excited due to intrinsic phase noise in lasers.

**C. Stronger Feedback Strength ($\kappa = 2000 \tau_{\text{f}}^{-1}$)**

In the experiments, when the feedback strength is stronger, random sinusoidal burst and two operation modes were observed. The frequency of large-amplitude sinusoidal oscillation and the frequency separation of the two observed modes coincide.

We first increased the coupling strength to $2000 \tau_{\text{f}}^{-1}$ and turned off the phase noise. A sinusoidal waveform was easily obtained. An example is shown in Fig. 12. From the calculated power spectrum of the sinusoidal waveform, the frequency is determined to be about $620 \tau_{\text{f}}^{-1}$. Just as the large-amplitude sinusoidal waveform observed in the experiment with stronger feedback, the amplitude of the sinusoidal waveform is several times larger than the fluctuation amplitude of the noise-driven relaxation oscillation and the frequency of the sinusoidal waveform is also several times larger than $f_{\text{RO}}$. When we increased $\kappa$, the frequency of the sinusoidal waveform also increased. This means that when coupling strength increases another deterministic state appears, sustaining large-amplitude sinusoidal oscillation and replacing the RO state.

![Fig. 12. (a) Numerical temporal waveform of $S(t)$ and (b) its power spectrum for $\kappa = 2000 \tau_{\text{f}}^{-1}$, where the phase noise is removed, i.e., $D_{\phi 0} = 0$.](image)

![Fig. 13. Determination of external cavity modes based on a graphical solution of (5) assuming $C = 3.65$, $\omega = 1.77 \times 10^{15} \text{rad/s}$, $\alpha = 1$, and $T = 5.6 \times 10^{-9} \text{s}$. The solid circles denote stable solutions and the empty circle denotes the unstable solution. The frequency difference between the two stable solutions is 13 MHz.](image)

Apparently, the experimentally observed random sinusoidal burst is a kind of random switch between the two states, i.e., the sustained large-amplitude sinusoidal oscillation and noise-driven relaxation oscillation. However, in numerical simulations, we cannot reproduce the observed bursts. To be more specific, only the degradation of the sinusoidal waveforms were numerically obtained and no switching was achieved even with an increase of the phase noise. This behavior is totally different from the numerical result of random chaotic burst in the weakly-coupled regime, in which clear switching takes place.

Although a direct numerical verification has not been achieved, some analytical pictures provide the mechanism. As mentioned above, when $C < 1$, there is only one external cavity mode solution, while for $C > 1$ multiple external cavity mode solutions appear. In the experiments, in the case of stronger feedback with $\kappa = 5.5 \times 10^7 \text{s}^{-1}$, $C \sim 3.65 > 1$. Multiple solutions are expected in such a stronger feedback condition. Since (5) is a transcendental equation, we solved it graphically with the following parameters: $C = 3.65$, $\omega = 1.77 \times 10^{15} \text{rad/s}$, $\alpha = 1$, and $T = 5.6 \times 10^{-9} \text{s}$. Based on the stability analysis of (6), the solutions of (5) are determined graphically as shown in Fig. 13, in which $\Delta \omega / 2\pi$ is the optical frequency shift. Referring to Fig. 13, the solid circles denote the external cavity mode solutions while the empty ones denotes the saddle (antimode). The oscillation frequency separation of the two stable external cavity mode solutions is about 13 MHz, which is close to the experimental result. This implies that the two observed modes measured by the scanning Fabry–Perot interferometer are nothing other than the external cavity modes described above. From (5), it is also easy to show that the frequency spacing of the two stable external cavity modes increases with an increase of the coupling strength as well as the delay time. The value ranges from 0, for single-mode operation, to $c/2\pi L$, for very strong coupling strength.

When multiple external cavity modes were allowed to exist in the laser system, mode hopping between these external cavity modes is expected to happen. Furthermore, complex nonlinear dynamic behavior due to the interaction between these external cavity modes was also expected. Therefore, large-amplitude
sinusoidal wave generation is considered to arise through the nonlinear interaction between these two external cavity modes. There are two possibilities which may explain random switching of this sinusoidal waveform observed in the experiment. One is the noise-induced switching between a one-external cavity mode state and a two-external cavity mode state, in which a sinusoidal waveform appears. Another explanation is that the switching occurs “deterministically,” reflecting the self-induced fluctuation of the external cavity length through the intensity-dependent self-phase modulation in the fiber in the strongly coupled regime, which is not included in (2)–(4).

V. Conclusion

In our experiments, two kinds of instabilities were observed in an LD-pumped microchip Nd:YVO₄ single-mode laser with fiber feedback in different feedback strength regimes. Random switching between noise-driven relaxation oscillation and chaotic spike oscillation was observed in the weaker feedback regime, \( \kappa = 1.1 \times 10^5 \text{ s}^{-1} \text{–} 2.5 \times 10^7 \text{ s}^{-1} \). In this regime, the main frequency component in the power spectrum was shifted to a lower value and the lasing linewidth broadened as chaotic spiking appeared. In LDs with external feedback, linewidth broadening or narrowing were observed under certain experimental conditions. However, this linewidth variation is very sensitive to the effective cavity length \( L \). Here, we focused only on what happened to the laser linewidth when instabilities occurred and only linewidth broadening was observed with a concurrent appearance of chaotic spike oscillation. Employing the single-mode Lang–Kobayashi equations, the random-switching waveform between noise-driven relaxation oscillation and chaotic spike oscillation as well as the power spectrum frequency shift and linewidth broadening have been reproduced successfully by including a relevant phase noise in the Nd:YVO₄ laser.

In a stronger feedback regime, \( \kappa = 2.5 \times 10^7 \text{ s}^{-1} \text{–} 1.7 \times 10^8 \text{ s}^{-1} \), the random switching between noise-driven relaxation oscillation and large-amplitude sinusoidal oscillation was observed. Two operational modes appear randomly in the oscillation spectrum and the oscillation frequency difference between the two observed modes almost coincide with the frequency of large-amplitude sinusoidal oscillation. In contrast to the weakly-coupled regime, we did not observe any significant linewidth broadening or narrowing in this regime. The sinusoidal waveform with a frequency value approximately equal to the oscillation frequency spacing between two stable external cavity modes was confirmed by graphical solutions of external cavity modes and numerical simulations of the single-mode Lang–Kobayashi equations together with large coupling coefficient and zero phase noise. However, with the present parameter setting, we cannot reproduce the sinusoidal burst. Further investigations over a wider region of parameters and the effect of intensity-dependent self-phase modulation in the optical fiber are under progress.

The linewidth enhancement factor, \( \alpha_o \), is an important parameter here. Although there is no experimental measurement to determine the values in microchip solid-state lasers, due to the substantially large \( \chi^2 \) nonlinearity of Nd:YVO₄ itself, as shown in [31], we assumed a nonzero value in simulations and obtained good agreement with the experiment.

Finally, we would like to emphasize that the observed instabilities are totally different from the instabilities in LDs. The phase-noise-driven chaotic spiking instability occurred in a very weak feedback strength regime, which is thought to be stable in LDs. In this regime, phase noise plays an important role. In the stronger feedback regime, random sinusoidal burst, which as to our knowledge, has also not been observed in LDs. It is worthwhile to note that the delay time due to optical feedback is larger than the characteristic time scale of microchip solid-state lasers. However, for the case of LDs, the delay time is usually smaller than the characteristic time scale of LDs. The peculiar differences in time scales bring out unique dynamical features for solid-state lasers with optical feedback which are significantly different from that of LDs with optical feedback.

REFERENCES


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