Graph Factors and Matching Extensions

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Chapter 1

Matchings and Perfect Matchings

As a young branch of mathematics, Graph Theory has experienced the explosion growth as the same phenomenon was taking place in computing science and communication networking. In the mean time, there are many new terminologies and knowledge accumulated in the process. So naturally, often there are more than one names or notions defined for a same entity. So we start to list the most terms and notions used in this book and hope to provide the readers with a consistent reference.

Notions

Let G be a graph, with vertex set $V(G)$ and edge set $E(G)$. Let f be a positive integer-valued function defined on vertex set $V(G)$.

$|S|$: order of a set S ;

$\lfloor x \rfloor$: the largest integer not greater than x ;

$\lceil x \rceil$: the smallest integer not less than x ;

$V(G)$: the vertex set of G , sometimes, we use $|G|$ for the order of G , i.e., $|G| = |V(G)|$;

$E(G)$: the edge set of G ; occasionally we write $||G||$ for the size of G , i.e., $||G|| = |E(G)|$;

$A \triangle B$: symmetric difference of two sets A and B , i.e., $A \triangle B = (A - B) \cup (B - A)$;

$d_G(v)$: the degree of a vertex v in G ;

$\delta(G)$: the minimum degree of G ;

$\Delta(G)$: the maximum degree of G ;

$\nabla(v)$: the edges incident with the vertex v ;

$\nabla(S)$: the set of edges with exactly one end-vertex in the set S ;

$N_G(S)$: the set of vertices adjacent to a vertex of S , i.e., $N_G(S) = \{x \mid xy \in E(G) \text{ and } y \in S\}$;

$E_G(S, T)$: the edges with one end in S and another end in T and $e_G(S, T) = |E_G(S, T)|$;

- $\sigma_k(G)$: degree sum of k independent vertices;
- $\mu(G)$: the matching number or fractional matching number of G ;
- $\eta(G)$: maximum cardinality of a join of G ;
- $\xi(\Sigma)$: the smallest integer such that no graph G embeddable in surface Σ is $\xi(\Sigma)$ -extendable;
- $ext(G)$: extendability number of a graph G ;
- $b(G)$: the number of bricks of G in a decomposition;
- $\hat{c}(G)$: the number of critical $\{K_2, K_3\}$ -components in G ;
- $p(G)$: the number of Petersen graphs;
- $i\mu(G)$: the induced matching number;
- G^c : the complement of G ;
- $G[S]$: the induced subgraph of G by a vertex set S ;
- $o(G)$: the number of odd components of G ;
- $\omega(G)$: the number of components of G ;
- $c(G)$: the circumference of G ;
- $\beta^*(G)$: the vertex covering number of G ;
- $I(G)$: the set of isolated vertices in G ;
- $i(G)$: the number of isolated vertices in G ;
- $D(G)$: the set of all vertices in G which are not saturated by at least one maximum matching of G ;
- $A(G)$: the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$;
- $C(G) = V(G) - D(G) - A(G)$;
- $fc(G)$: the number of factor-critical components in G ;
- $tc(G)$: the number of triangle clusters in G ;
- $oc(G - S)$: the number of those components of $G - S$ that are odd cacti;
- $\Phi(G)$: the number of 1-factors in G ;
- G_δ : the subgraph induced by the vertices with minimum degree in G ;
- G_Δ : the subgraph induced by the vertices with maximum degree in G ;
- $\kappa(G)$: the vertex connectivity of G ;
- $\kappa'(G)$: the edge connectivity of G ;
- $\lambda(C)$: characteristic of a cut C ;
- $c\lambda(G)$: the cyclic connectivity of G ;
- $\alpha(G)$: independent number of G ;
- $\gamma(G)$: the dominating number;
- $\gamma^*(G)$: the genus of G ;
- $\chi'(G)$: the edge chromatic number of G ;
- $\chi'_c(G)$: the edge-cover number of G ;
- $\tau_2(G)$: the size of a minimum 2-covering of G ;
- $G \vee H$: the *join* of two graphs G and H ;
- G^r : the r th power of a graph G .

$t(G)$: toughness of G introduced by Chvátal [153],

$$t(G) = \min\left\{\frac{|S|}{\omega(G-S)} \mid S \subseteq V(G), \omega(G-S) \geq 2\right\}$$

$\tau(G)$: a variation of toughness introduced by Enomoto [187],

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S) - 1} \mid S \subseteq V(G), \omega(G-S) \geq 2\right\}$$

We define, for all disjoint subsets S and T of $V(G)$,

$$\delta_G(S, T) = \begin{cases} k|S| + \sum_{x \in T} d_G(x) - k|T| - e_G(S, T) - q_G(S, T) & \text{for } k\text{-factor;} \\ f(S) + \sum_{x \in T} d_G(x) - f(T) - e_G(S, T) - q_G(S, T) & \text{for } f\text{-factor;} \\ f(S) + \sum_{x \in T} d_G(x) - g(T) - e_G(S, T) - q_G(S, T) & \text{for } (g, f)\text{-factor.} \end{cases}$$

1.1 Definitions and terminologies

To understand the developments of matchings and factors in graphs, we would like to review the history briefly by listing the most significant milestones in this fascinating area of graph theory.

The first attempt on the study of factors was due to Danish mathematician Petersen (1891), who proved that every graph of even degrees can be decomposed into the union of edge-disjoint 2-factors. This was motivated from the study of an algebraic factorization problem. He also showed that every 2-connected cubic graph has a 1-factor. These two results can be viewed as a forerunner of modern graph factor theory.

For matchings in bipartite graphs, König (1931) and Hall (1935) obtained the so-called König-Hall Theorem (sometimes, it is known as Hall's Theorem). Due to its wide applications to many graph theory problems and to other branches of mathematics, König-Hall Theorem remains one of most influential graph-theoretic results.

In 1947, Tutte gave a characterization (i.e, so-called Tutte's 1-Factor Theorem) for the existence of perfect matchings in arbitrary graphs and it has become a cornerstone of factor theory. Till now, this elegant theorem is still one of the most fundamental results in factor theory. Subsequently, Tutte (1952) extended the techniques in the proof of 1-Factor Theorem to obtain a sufficient and necessary condition for a graph to have an f -factor.

Gallai (1964) and Edmonds (1965), independently, investigated a canonical decomposition of arbitrary graphs in terms of its maximum matchings and thus revealed the structure of graphs related to matchings. This deep and important result is referred as Gallai-Edmonds Structure Theorem. The existence of the canonical decomposition

and an efficient method to find such a decomposition allowed Edmonds to obtain the first polynomial algorithm for finding maximum matchings in graphs.

The most general degree-constrained factors, (g, f) -factors, were studied by Lovász (1970). He gave necessary and sufficient conditions for a graph to have a (g, f) -factor. This theorem generalized the criteria of all other factors, such as 1-factors, k -factors, f -factors and $[a, b]$ -factors.

For the more comprehensive account of history on matching theory and graph factors, readers can refer to Preface of Lovász and Plummer's *Matching Theory* or Biggs, Lloyd and Wilson's *Graph Theory 1736-1936*.

In this book, we will mainly deal with factors in *finite undirected simple* graphs. Some results also hold for graphs with multiple edges, in such cases, we will point out.

The *degree* of a vertex v in a graph G , denoted by $d_G(v)$, is the number of edges of G incident with v , each loop counting as two edges. In particular, if G is a simple graph, $d_G(v)$ is the number of neighbours of v in G . A vertex of degree zero or one are called an *isolated vertex* and a *leaf*, respectively. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G .

A *cut-edge* of a graph is one whose deletion results in one more component. Sometimes, a cut-edge is also called a *bridge*.

A *matching* M of a graph G is a subset of $E(G)$ such that any two edges of M have no end-vertices in common. A matching of k edges is called a *k -matching*. Let d be a non-negative integer. A matching is called a *defect- d matching* if it covers exactly $|V(G)| - d$ vertices of G . A defect-0 matching is called a *perfect matching* and defect-1 matching is called *near-perfect matching*.

A perfect matching is also referred as 1-*factor* since it is a 1-regular subgraph. If a graph has 1-factors, it is called 1-*factorable*. There are probably equal numbers of people using “perfect matchings” rather than “1-factors”, so we will use them non-discriminatively.

If an edge e is contained in a 1-factor, we called e an *allowed edge*. If an edge f does not lie in any 1-factor, then f is *forbidden*.

A *covering* of a graph G is a subset S of $V(G)$ such that every edge of G has at least one end in S . A covering S^* is a *minimum covering* if G has no covering S with $|S| < |S^*|$. The order of a minimum covering is called *covering number* and denoted by $\beta^*(G)$.

A graph G is said to be *bicritical* if for every pair of distinct vertices u and v , $G - \{u, v\}$ has a 1-factor. A graph G is *factor-critical* if $G - u$ has a 1-factor for every $u \in V(G)$.

A graph G is called *n -factor-critical* if the subgraph $G - S$ has a 1-factor for all n -subset S of $V(G)$.

Clearly, the concept of n -factor-criticality is a generalization of bicriticality and factor-criticality. That is, bicritical graphs and factor-critical graphs are n -factor-

Figure 1.1: (a) a factor-critical graph and (b) a bicritical graph

critical graph when $n = 1$ and 2 , respectively.

From the definitions, we can see the following result easily.

Proposition 1.1.1 (a) A graph G is factor-critical if and only if the join of G and K_1 , $G \vee K_1$, is bicritical;

(b) a graph G is bicritical if and only if $G - v$ is factor-critical for every $v \in V(G)$.

Let G be a connected graph with a 1-factor and $|V(G)| \geq 2k + 2$. If each k -matching of G is contained in a 1-factor, we call G a k -extendable graph. For convenience, a 0-extendable graph means a graph which has a 1-factor.

Clearly bicritical graphs are 1-extendable. A 3-connected bicritical graph is called a *brick*. A *brace* is a 2-extendable bipartite graph (i.e., 3-connected 2-extendable bipartite graph).

Figure 1.2: (a) a brick and (b) a brace

Factors in a graph is spanning subgraphs satisfying some given properties. A k -factor of a graph G is a spanning subgraph H such that $d_H(x) = k$ for every $x \in V(G)$.

Let f be an integer-valued function defined on $V(G)$, i.e.,

$$f : V(G) \rightarrow \mathbb{Z}^+ = \{0, 1, 2, \dots\},$$

then a spanning subgraph H such that $d_H(x) = f(x)$ for all $x \in V(G)$ is called an f -factor.

A more general factor is so-called (g, f) -factors: for two functions $g, f : V(G) \rightarrow \mathbb{Z}$, a spanning subgraph H is called a (g, f) -factor if $g(x) \leq d_H(x) \leq f(x)$ for all $x \in V(H)$. In addition, if $g(x) \equiv f(x) \pmod{2}$ for all $x \in V(G)$, a (g, f) -factor F with $d_F(x) \equiv f(x)$ is called a (g, f) -parity-factor.

Figure 1.3: When $g(x) = 1$, $f(x) = 3$, a spanning subgraph H is a (g, f) -factor

For two positive integers a, b , let $g(x) = a$ and $f(x) = b$ for all vertex $x \in V$, then (g, f) -factors are called $[a, b]$ -factors.

A set $S \subset V$ is a *dominating set* of G if every vertex in V is either in S or is adjacent to a vertex in S . If A and B are subsets of V , we say that A *dominates* B if every vertex of B has a neighbor in A or is a vertex of A ; we also say that B is *dominated* by A . The *dominating number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set. A set S is *2-dominating* if every vertex of V is either in S or is within distance 2 to S .

We only define the most often used terms of the book in this section. For other undefined terms, we follow Bollobás [85].

1.2 Matchings in bipartite graphs and augmenting path

In this section, we focus on matchings and perfect matchings in bipartite graphs. From historic point of view, it is the starting point of matching theory. This is not surprise since matchings in bipartite graphs are closely related to other disciplines of mathematics (e.g., matrix theory and set theory) and to many application problems (e.g., the job assignment problem). Furthermore, the properties and techniques of matchings in bipartite graphs also motivate the research of matchings in non-bipartite graphs.

For a matching M , a vertex x of G is called *saturated* or *covered* by M if x is incident to an edge of M . A matching M is called a *maximum matching* if $|M| \geq |M'|$ for any matching M' of G . Clearly, a perfect matching (i.e., a 1-factor) is a maximum matching saturating every vertex of a graph. Of course, any graph with perfect matchings must have even number of vertices.

In many applications, one wishes to find a matching in a bipartite graph $G =$

(X, Y) which covers every vertex in X . Necessary and sufficient conditions for the existence of such a matching were proven by Hall (1935). To this day, Hall's Theorem remains one of most applied graph-theoretic results. The proof here is due to Rado.

Theorem 1.2.1 (Hall's Theorem, Hall (1935)) *Let $G = (X, Y)$ be a bipartite graph. Then G has a matching saturating all vertices of X if and only if for any $S \subseteq X$,*

$$|N(S)| \geq |S|. \quad (1.1)$$

Proof. The necessity is clear.

Let G be a minimal bipartite graph satisfying the condition (1.1). It suffices to show that G consists of $|X|$ independent edges.

Otherwise, G contains two edges, x_1y and x_2y , where $x_1, x_2 \in X$, $x_1 \neq x_2$ and $y \in Y$. From the minimality, the deletion of either of these edges invalidates (1.1). Thus there are two subsets $X_1, X_2 \subset X$ such that for $i = 1, 2$ we have $|N(X_i)| = |X_i|$, and x_i is the only vertex of X_i adjacent to y . Then

$$\begin{aligned} |N(X_1) \cap N(X_2)| &\geq |N(X_1 - \{x_1\}) \cap N(X_2 - \{x_2\})| + 1 \\ &\geq |N(X_1 \cap X_2)| + 1 \geq |X_1 \cap X_2| + 1. \end{aligned}$$

But this implies the following contradiction:

$$\begin{aligned} |N(X_1 \cup X_2)| &= |N(X_1) \cup N(X_2)| \\ &= |N(X_1)| + |N(X_2)| - |N(X_1) \cap N(X_2)| \\ &\leq |X_1| + |X_2| - |X_1 \cap X_2| - 1 \\ &= |X_1 \cup X_2| - 1. \end{aligned}$$

□

Note that Hall's Theorem also holds for multigraphs. The above result is the usual version of matching characterization in bipartite graphs. However, we present the next version in terms of isolated vertices, which has a strong similarity to Tutte's 1-Factor Theorem appeared later. Recall that $i(G)$ denotes the number of isolated vertices in G .

Theorem 1.2.2 (Hall's Matching Theorem, Hall (1935)) *Let $G = (X, Y)$ be a bipartite graph. Then G has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$,*

$$i(G - S) \leq |S|.$$

Remark 1.2.1. For every bipartite graph $G = (X, Y)$, with $|X| \geq |Y|$ and maximum degree Δ , König (1916) showed that there exists Δ -regular bipartite super-graph $G' = (X, Y')$ (a multigraph is allowed) such that $Y \subseteq Y'$ and G is a subgraph of G' .

Corollary 1.2.1 (König (1916)) *Every r -regular bipartite graph G can be decomposed into r edge-disjoint 1-factors. In particular, G has a 1-factor.*

Corollary 1.2.2 *Every bipartite graph G has a maximum matching which covers all the vertices of maximum degree.*

Proof. By Remark 1.2.1, there exists a $\Delta(G)$ -regular supergraph G' containing G . From Corollary 1.2.1, G' has a 1-factor and so G has a matching covers all the vertices of maximum degree. In turn, this matching can be augmented to a maximum matching of G . $\square$

A family of graphs which share many properties with bipartite graphs was discovered by Fulkerson, Hoffman and McAndrew [221]. A graph G is said to have the *odd cycle property* (OCP in short) if every pair of odd cycles in G either have a vertex in common or are joined by an edge. We shall discuss the relation between this property and the existence of factors in much more details in Section 2.2.

The following is a generalization of Corollary 1.2.1.

Theorem 1.2.3 *If G is a r -regular graph of even order with the odd cycle property, then G has a 1-factor.*

Let M be a matching in a graph G . An M -alternating path or cycle in G is a path or cycle whose edges are alternately in M and $E(G) - M$. Note that an M -alternating path might or might not start or end with edges of M . If neither its origin nor its terminus is covered by M , the M -alternating path is called an M -augmenting path.

Figure 1.4: An M -alternating cycle $C = ablk$, an M -alternating path $P = defghi$ and an M -augmenting path $P' = cdefghij$.

The idea of M -augmenting path is first introduced by Berge (1957). Today, it has become a very matured and standard technique in the study of factor theory. There are many variations of augmenting path defined for different problems, we

shall see all kinds of variations in the coming chapters. It is not an overstatement that augmenting path technique is the most powerful technique not only in the study of matchings and also for the study of all factors.

Berge (1957) use augmenting path to give a characterization of maximum matching.

Theorem 1.2.4 (Berge (1957)) *A matching M in a graph G is maximum if and only if G has no M -augmenting path.*

Proof. Let M be a matching in G . Suppose that G contains an M -augmenting path P . Then $M' := M \triangle E(P)$ is a matching in G , and $|M'| = |M| + 1$. Thus M is not a maximum matching.

Conversely, suppose that M is not a maximum matching, and let M'' be a maximum matching in G , so that $|M''| > |M|$. Set $H := G[M \triangle M'']$. Then each vertex of H has degree one or two in H , for it can be incident with at most one edge of M and one edge of M'' . Consequently, each component of H is either an even cycle with edges alternately in M and M'' , or else a path with edges alternately in M and M'' .

Because $|M''| > |M|$, the subgraph H contains more edges of M'' than of M , and therefore some path-component P of H must start and end with edges of M'' . The origin and terminus of P , being covered by M'' , are not covered by M . The path P is thus an M -augmenting path in G , a contradiction. $\square$

Hall's Theorem can be restated in a more general version. We denote the matching number, the size of a maximum matching, by $\mu(G)$.

Theorem 1.2.5 *Let $G = (X, Y)$ be a bipartite graph, M a matching in G , and U the set of M -unsaturated vertices in X . Then*

- (a) *for any subset S of X , $|U| \geq |N(S)| - |S|$; and*
- (b) *$|U| = |N(S)| - |S|$ if and only if M is a maximum matching of G .*

Furthermore, the matching number of G is given by

$$\mu(G) = |X| - \max\{|S| - |N(S)| \mid S \subseteq X\}.$$

The expression for $\mu(G)$ is known as the König-Ore Formula.

Next, we investigate the properties of matching transformation in bipartite graphs. That is, given two matchings, we can start with one and transform gradually to another one through the operation of symmetric difference.

Theorem 1.2.6 *A maximum matching M of a graph G can be obtained from any other maximum matching M' by a sequence of transfers along alternating cycles and paths of even lengths.*

Proof. From the proof of Theorem 1.2.4, every component of $G[M' \triangle M]$ is either an alternating cycle or an alternating even path relative to M' . Changing M' in each component in turn will transform M' into M . $\square$

Theorem 1.2.7 (Dulmage and Mendelsohn [438]) *Let $G = (X, Y)$ be a bipartite graph and M_1, M_2 be two matchings in G . Then there is a matching $M \subseteq M_1 \cup M_2$ such that M covers all the vertices of X saturated by M_1 and all the vertices of Y saturated by M_2 , that is, $V(M) \supseteq (V(M_1) \cap X) \cup (V(M_2) \cap Y)$.*

Proof. Let X_i (respectively, Y_i) be the vertices of X (of Y) covered by M_i ($i = 1, 2$). Let $G_1, G_2, \dots, G_k$ be the connected components of $G[M_1 \triangle M_2]$. Then each G_i is an even cycle or path. Let $M_{1i} = E(G_i) \cap M_1$ and $M_{2i} = E(G_i) \cap M_2$ ($1 \leq i \leq k$) and define a matching P_i in each G_i :

$$P_i = \begin{cases} M_{1i} & \text{if } G_i \text{ is a cycle;} \\ M_{1i} & \text{if there is a vertex } v \in V(G_i) \cap (X_1 - X_2); \\ M_{2i} & \text{if there is a vertex } v \in V(G_i) \cap (Y_2 - Y_1). \end{cases}$$

It is not hard to check that $M = (M_1 \cap M_2) \cup P_1 \cup P_2 \cup \dots \cup P_k$ is the required matching. $\square$

The local neighborhood condition can be used as a sufficient condition for the maximum matching in bipartite graphs. The following theorem is due to Alon.

Theorem 1.2.8 *Let $G = (X, Y)$ be a bipartite graph. If one of the following conditions holds,*

- (a) *$d(x)$ is odd for all $x \in X$ and any two vertices of X have an even number of common neighbors; or*
- (b) *$d(x) \geq 1$ for all $x \in X$ and $d(x) \geq d(y)$ for all $xy \in E(G)$, where $x \in X$ and $y \in Y$,*

then G has a matching saturating X .

The next result is of type of degree condition.

Theorem 1.2.9 *Let $G = (X, Y)$ be a bipartite graph with $|X| = |Y| = n$. If $\delta(G) \geq \frac{n}{2}$, then G has a 1-factor.*

A matching is *induced* if no two edges in the matching are joined by an edge in G . Unlike the usual matching problem, the induced matching is a much more difficulty problem to deal with. The problem of finding a maximum induced matching is *NP*-hard, even for bipartite graphs and for planar graphs of maximum degree 4 (see [111]). The polynomial algorithms for solving the maximum induced matching problem are found only for a few families of special graphs (e.g., weakly chordal graphs, circle

graphs, chordal graphs, and outerplanar graphs). However, we have a neat formula for the size of maximum induced matching, i.e., induced matching number $i\mu(G)$, in bipartite graphs.

Theorem 1.2.10 (Liu and Zhou [381]) *Let $G = (X, Y)$ be a connected bipartite graph. Then the size of a maximum induced matching M of G is*

$$i\mu(G) = \max\{|S| \mid S \subseteq X \text{ such that } N_G(T) \neq N_G(S) \text{ for all } T \subset S\}.$$

Matchings in bipartite graphs and Hall's Theorem have proved to be a valuable tool both in graph theory and in other areas of mathematics. In fact, Hall's Theorem has several equivalent formulations, including the following one in terms of set systems.

Let $\mathcal{A} = (A_i : i \in I)$ be a finite family of (not necessarily distinct) subsets of a finite set A . A *system of distinct representatives* (SDR) for the family $\mathcal{A}$ is a set $\{a_i \mid i \in I\}$ of distinct elements of A such that $a_i \in A_i$ for all $i \in I$. In this language, Hall's Theorem says that $\mathcal{A}$ has a system of distinct representatives if and only if $|\cup_{i \in J} A_i| \geq |J|$ for all subsets J of I . To see that this is indeed a reformulation of Hall's Theorem, let $G = (X, Y)$, where $X = I$, $Y = A$, and $E(G) = \{ia \mid a \in A_i, \text{ where } i \in I \text{ and } a \in A\}$.

The other formulations of Hall's Theorem can be found in $(0, 1)$ -matrix theory and transversal matroid.

1.3 Matchings in non-bipartite graphs

In this section, we consider the existence of 1-factors in arbitrary graphs. The characterization of 1-factors was given by Tutte in 1947. It is one of most important theorems in graph theory for several reasons: firstly 1-factor as a 1-regular spanning subgraph is a simplest but the most essential structure of any graph; secondly due to its wide applications in other areas of mathematics and in optimization problems (it often serves as subroutine in many optimization algorithms); finally it is one of earliest topics in graph theory which has a relative complete understanding and matured techniques comparing with other topics in graph theory.

A component C of a graph is called *odd* if $|V(C)|$ is odd; otherwise, it is called *even*. Recall that $o(H)$ is the number of odd components in a graph H . Using a parity argument, it is not hard to see that

$$o(G - S) + |S| \equiv |V(G)| \pmod{2}. \quad (1.2)$$

If G is of even order, then $|V(G)| \equiv 0 \pmod{2}$ and thus

$$o(G - S) \equiv |S| \pmod{2}. \quad (1.3)$$

This fact is quite handy when estimating the number of odd components in a graph.

Now we present Tutte's criterion for 1-factors. Besides Tutte's original proof for this theorem, there are several shorter and simpler proofs, respectively, from Anderson (1971), Lovász (1975), as well as Ore's transformation from bipartite graphs. Here we use the proof from Lovász.

Theorem 1.3.1 (Tutte's 1-Factor Theorem, Tutte (1947)) *A graph G has a 1-factor if and only if for any $S \subseteq V(G)$*

$$o(G - S) \leq |S|. \quad (1.4)$$

Proof. *Necessity.* Since the odd components of $G - S$ must have vertices matched to distinct vertices of S , we have $o(G - S) \leq |S|$.

Sufficiency. Note that adding an edge joining two components of $G - S$ does not increase the number of odd components. Hence (1.4) is preserved from the addition of edges. That is, let $G' = G \cup e$ and $S \subseteq V(G)$, then $o(G' - S) \leq o(G - S) \leq |S|$. Furthermore, if G' has no 1-factor, then G has no 1-factor either.

Assume that G satisfies (1.4) but has no 1-factor. Without loss of generality, let G be (edge-wise) maximal such a graph. Note that (1.4) implies that $|V(G)|$ is even (considering $S = \emptyset$).

Let S_0 be the set of vertices in G that have degree $|V(G)| - 1$.

Case 1. $G - S_0$ consists of disjoint complete graphs.

In this case, vertices in each component of $G - S_0$ can be paired in any way, with one extra in the odd components. Since $o(G - S_0) \leq |S_0|$ and each vertex of S_0 is adjacent to all of $G - S_0$, we can match the leftover vertices to vertices of S_0 .

The remaining vertices are in S_0 , which is a clique. Since $V(G)$ is even, by parity, an even number of vertices remain in S_0 and thus we can form a 1-factor.

Case 2. $G - S_0$ is not a disjoint union of cliques.

In this case, $G - S_0$ has two vertices at distance 2; these are nonadjacent vertices x, z with a common neighbor $y \notin S_0$. Furthermore, $G - S_0$ has another vertex w not adjacent to y , since $y \notin S_0$. By the maximality of G , adding an edge to G creates a 1-factor; let M_1 and M_2 be 1-factors in $G \cup xz$ and $G \cup yw$, respectively. It suffices to show that $M_1 \triangle M_2$ contains a 1-factor avoiding xz and yw , because this will be a 1-factor of G .

Let $F = M_1 \triangle M_2$. Since $xz \in M_1 - M_2$ and $yw \in M_2 - M_1$, both xz and yw are in F . Since every vertex of G has degree 1 in each M_1 and M_2 , every vertex of G has degree 0 or 2 in F . Hence the components of F are even cycles and isolated vertices.

Let C be the cycle of F containing xz (see Figure 1.3). If C does not also contain yw , then the desired 1-factor consists of the edges of M_2 from C and all of M_1 not

Figure 1.5: Alternative Paths

in C . If C contains both yw and xz , then to avoid them we use yx or yz . In the portion of C starting from y along yw , we use edges of M_1 to avoid using yw . When we reach $\{x, z\}$, we use zy if we arrive at z ; otherwise, we use xy . In the remainder of C we use the edges of M_2 . We have produced a 1-factor of C that does not use xz or yw . Combined with M_1 or M_2 outside C , we have a 1-factor of G . $\square$

Hall's Theorem and Tutte's 1-Factor Theorem are in fact equivalent. They both belong to a large family of fundamental results in graph theory which are all equivalent to each other (see Jacobs and Heiratssatz [266]).

As one of applications of 1-Factor Theorem and also a refinement, Theorem 1.3.1 can also be restated in terms of factor-critical graphs. This result was a by-product of the algorithm for finding a maximum cardinality matching by Edmonds [168]. A non-algorithmic proof of this theorem was given by Anderson [55].

Let $fc(G)$ be the number of factor-critical components of graph G .

Theorem 1.3.2 (Edmonds [168]) *A graph G of even order has a 1-factor if and only if for every subset $S \subseteq V(G)$,*

$$fc(G - S) \leq |S|.$$

Proof. Since every factor-critical component is an odd component, so the necessity follows immediately from 1-Factor Theorem.

Conversely, to the contrary, assume that G has no 1-factor. Then, by 1-Factor Theorem, there exists a subset $S \subseteq V(G)$ such that $o(G - S) > |S|$ and $S \neq \emptyset$ or $V(G)$. Note the fact

$$o(G - S') \leq |S'| \quad \text{for all } S_0 \subset S' \subseteq V(G). \quad (1.5)$$

Choose *maximal* such a set S_0 . Then the maximality of S_0 implies that $G - S_0$ has no even component. Next we show that every component of $G - S_0$ is factor-critical, i.e., $fc(G - S_0) = o(G - S_0)$.

Let C be an odd component of $G - S_0$, and v any vertex of C . Then for every $T \subseteq V(C - v)$, (1.5) implies

$$\begin{aligned}
|S_0| + 1 + |T| &\geq o(G - (S_0 \cup \{v\} \cup T)) \\
&= o(G - S_0) - 1 + o(C - (\{v\} \cup T)) \\
&> |S_0| - 1 + o((C - v) - T).
\end{aligned}$$

Hence $o((C - v) - T) < |T| + 2$, which implies $o((C - v) - T) \leq |T|$ by (1.2). Therefore $C - v$ has a 1-factor by 1-Factor Theorem, or C is factor-critical. Consequently, $fc(G - S_0) = o(G - S_0) > |S_0|$, a contradiction to the assumption of the theorem. $\square$

In fact, Tutte's 1-Factor Theorem can be refined further by considering only the special subsets S (i.e., barrier set) and the number of factor-critical components. More refinement, i.e., Gallai-Edmonds Structure Theorem, will be discussed in Section 1.4.

When applying 1-Factor Theorem to various classes of graphs, the criterion can be much simpler and nicer. The next result is such an example and it can be proven easily by induction.

Proposition 1.3.1 (Chungphaisan (1976)) *A tree T of even order has a 1-factor if and only if $o(T - v) = 1$ for every vertex v of T .*

Augmenting path (or chain) technique lies in the heart of factor theory. It is an effective way to adjust a subgraph to get a better subgraph (or factor). Its significance is also reflected in the fact that if an augmenting technique is developed for a given factor problem (e.g., 1-factors, k -factors, f -factors, L -factors with $(*)$ property, etc.), then it often can be converted into a polynomial algorithm for the problem.

Augmenting path (or chain) technique possesses many nice properties, as shown in the next theorem. These properties are important for the adjustment process to reach the optimal subgraphs. The uttermost usage of such properties can be found in treating L -factors in [406, 407].

For a matching M and a vertex set S , if $V(M) \cap S = \emptyset$, then we say that M avoids S .

Theorem 1.3.3 (see [13]) *Let A and B be vertex subsets of a graph G such that $|A| < |B|$. Then*

- (a) *if there exist two matchings, one saturating A and the other saturating B , then there exists a matching that saturates A and at least one vertex in $B - A$;*
- (b) *if there exist two maximum matchings, one avoiding A and the other avoiding B , then there exists a maximum matching that avoids A and at least one vertex in $B - A$.*

Proof. (a) Let M_A and M_B be matchings in G which saturate A and B , respectively. If M_A saturates one vertex in $B - A$, then M_A itself is the desired matching. Hence we may assume that M_A avoids $B - A$.

Consider the subgraph $G[M_A \Delta M_B]$. Then for every vertex b in $B - A$, there exists a path in $G[M_A \Delta M_B]$ starting with b , which may end with a vertex in $A - B$. Since $|A| < |B|$ and no path of $G[M_A \Delta M_B]$ ends with a vertex in $A \cap B$, there exists a path P in $G[M_A \Delta M_B]$ that starts with $b_1 \in B - A$ and ends with $x \notin A$. Since P is an M_A -alternating path connecting b_1 and x , $M_A \Delta E(P)$ is the desired matching in G , which saturates A and $b_1 \in B - A$.

(b) Let N_A and N_B be maximum matchings in G which avoid A and B , respectively. We may assume that N_A saturates $B - A$ since otherwise N_A itself is the desired matching. Then by the same argument as in (a), there exists a N_A -alternating path P in $G[N_A \Delta N_B]$ starting with a vertex $b \in B - A$ and ending with a vertex not contained in A . If both end-edges of P are contained in N_A , then $N_B \Delta E(P)$ is a matching with $||N_B|| + 1$ edges, contrary to the maximality of N_B . Thus one end-edge of P belongs to N_B , and thus $N_A \Delta E(P)$ is the desired maximum matching in G , which avoids A and $b \in B - A$. $\square$

To find a matching containing a given set of vertices, we have the following interesting result.

Proposition 1.3.2 (Kierstead [316]) *Let G be a graph and T the set of vertices of degree $\Delta(G)$. If $G[T]$ is a bipartite subgraph, then there exists a matching in G saturating all vertices of T .*

To conclude this section, we present a conjecture due to Macajova and Skoviera.

Conjecture 1.3.1 (Macajova and Skoviera (2005)) *Every 2-edge-connected cubic graph has two perfect matchings M_1 and M_2 such that $M_1 \cap M_2$ does not contain an odd edge-cut.*

1.4 Sufficient conditions for 1-factors

Although Tutte's 1-Factor Theorem provides a nice and simple criterion for 1-factors, to check the condition (1.4) is a non-trivial task. So many researchers have attempted to find simple sufficient conditions for the existence of 1-factors by using various graphic parameters, or to determine special classes of graphs to be 1-factorable. The graphic parameters having been studied include connectivity, claw-free, neighborhood, degree-sequence, toughness, binding number, etc. Those approaches not only provide a simple conditions for existence of 1-factors, and also enrich our understanding to the structures of graphs with 1-factors; more importantly it supplies many fresh ideas for the study of other factors. In fact, many of them have been generalized to k -factors, $[a, b]$ -factors and so on. We will examine it in more details in later chapters,

for instance, toughness in Sections 3.1 and 3.2, binding number in Section 3.3 and connectivity in Section 3.4.

When we use 1-Factor Theorem to show the existence of 1-factors or maximum matchings in a graph, often we have to deal with the number of odd components and counting the number of edges between odd components and a vertex set. Beside the relation (1.2), the following result will be handy for such estimations, in particular, in regular graphs.

Proposition 1.4.1 *Let G be a r -regular graph ($r \geq 2$), and $S \subseteq V(G)$. Then for each odd component C of $G - S$,*

$$e_G(C, S) \equiv r \pmod{2}$$

In particular, if G is an $(r - 1)$ -edge-connected r -regular graph, then $e_G(C, S) \geq r$.

Proof. Since $|C|$ is odd, we have

$$r \equiv r|C| = \sum_{u \in V(C)} d_G(u) = e_G(C, S) + 2|E(C)| \equiv e_G(C, S) \pmod{2}.$$

Moreover, if G is an $(r - 1)$ -edge-connected r -regular graph, then $e_G(C, S) \geq r - 1$, and by combining this inequality and parity, we have $e_G(C, S) \geq r$. $\square$

Next, we examine different parameter as sufficient condition for 1-factors.

Connectivity Condition:

Let us start with connectivity, which usually the first parameter one looks at, as a condition for the existence of 1-factors. The first significant result in this direction, actually the first on 1-factors in graphs, was obtained by Petersen (1891), who was particularly interested in 3-regular graphs.

Theorem 1.4.1 (Petersen [475]) *Let G be a 2-edge-connected 3-regular graph. Then G has a 1-factor.*

Proof. Let S be any nonempty proper subset of $V(G)$. Denote all the odd components of $G - S$ by $C_1, C_2, \dots, C_m$. From Proposition 1.4.1, we have $e_G(C_i, S) \geq 3$ for $1 \leq i \leq m$. Therefore

$$3|S| \geq e_G(C_1 \cup C_2 \cup \dots \cup C_m, S) \geq 3m.$$

Hence $|S| \leq m = o(G - S)$ or G has a 1-factor by 1-Factor Theorem. $\square$

Figure 1.6: A cubic graph without 1-factors

The connectivity condition in Theorem 1.4.1 is best possible, i.e., there exist connected cubic graphs which have no 1-factor. An example is shown in Figure 1.6.

This theorem can be generalized in several ways. Firstly, the 2-edge-connectivity condition can be replaced by “all the bridges of G are contained in the same path”; secondly, the theorem also holds for multigraphs as well; finally, of course, we can generalize it to general r -regular graphs. In fact, we prove a slightly stronger result by specifying a 1-factor to pass a given edge or to avoid other edges.

Theorem 1.4.2 *Let $r \geq 2$, and G be an $(r - 1)$ -edge-connected r -regular graph.*

- (a) *If $|V(G)| \equiv 0 \pmod{2}$, then, for every edge e of G , G has a 1-factor containing e . In particular, G has a 1-factor;*
- (b) *if $|V(G)| \equiv 1 \pmod{2}$, then, for every vertex v , $G - v$ has a 1-factor. That is, G is factor-critical.*

Proof. (a) Suppose, to the contrary, that there exists an edge $e = xy$ such that no 1-factor containing e . Then $G - \{x, y\}$ has no 1-factor. From 1-Factor Theorem, there exists $S' \subseteq V(G) - \{x, y\}$ such that $o(G - \{x, y\} - S') > |S'|$. Let $S = S' \cup \{x, y\}$. Then $o(G - S) \geq |S'| + 2 = |S|$ by parity. Let $C_1, C_2, \dots, C_m$ be the odd components of $G - S$, where $m = o(G - S)$. Then, by Proposition 1.4.1, we have $e_G(C_i, S) \geq r$. Hence

$$|\nabla(S)| \leq r|S| - 2 \leq rm - 2 \quad \text{since } e = xy \text{ is in } G[S]$$

and

$$|\nabla(S)| \geq e_G(C_1 \cup C_2 \cup \dots \cup C_m, S) \geq rm,$$

a contradiction. Hence, for every edge e of G , G has a 1-factor containing e .

(b) Suppose, to the contrary, that G is not factor-critical. Then, by Theorem ?? (in Section 6.1), there exists $\emptyset \neq S \subseteq V(G)$ such that $o(G - S) > |S| - 1$ or $o(G - S) \geq |S| + 1$ by parity. Let $C_1, C_2, \dots, C_m$ be the odd components of $G - S$, where $m = o(G - S)$. Then, by Proposition 1.4.1, we have $e_G(C_i, S) \geq r$. Hence

$$r|S| \geq |\nabla(S)| \geq e_G(C_1 \cup C_2 \cup \dots \cup C_m, S) \geq rm \geq r(|S| + 1)$$

a contradiction. Hence G is factor-critical. $\square$

The edge-connectivity condition in Theorem 1.4.2 is best possible, i.e., there exist infinitely many $(r - 2)$ -edge-connected r -regular graphs of even order that have no 1-factor. The above theorem holds for multigraphs as well. Note that in part (b) r must be an even integer.

A classic result due to Plesník (1974) showed that there exist 1-factors to avoid a set of edges in regular graphs with edge-connectivity condition. The proof is from [13].

Theorem 1.4.3 (Plesník [479]) *Let $r \geq 2$ be an even integer, and G be an $(r - 1)$ -edge-connected r -regular graph of even order. Then for any $r - 1$ edges $e_1, e_2, \dots, e_{r-1}$ of G , G has a 1-factor excluding $\{e_1, e_2, \dots, e_{r-1}\}$.*

Proof. Let $H = G - \{e_1, \dots, e_{r-1}\}$. We show that H has a 1-factor. For any vertex set $S \subseteq V(H) = V(G)$, let $C_1, C_2, \dots, C_m$ be the odd components of $H - S$, where $m = o(H - S)$. Then by the same argument as in the proof of Proposition 1.4.1, we have

$$e_G(C_i, V(G) - V(C_i)) \geq r - 1 \quad \text{and} \quad e_G(C_i, V(G) - V(C_i)) \equiv r \pmod{2}.$$

Hence

$$r \leq e_G(C_i, V(G) - V(C_i)) = e_G(C_i, S) + e_G(C_i, V(G) - S - V(C_i)).$$

Since C_i and C_j ($i \neq j$) are components of $H - S$, if there is an edge e between C_i and C_j , then $e \in \{e_1, \dots, e_{r-1}\}$. That is, $E_G(C_i, V(G) - S - V(C_i)) \subseteq \{e_1, \dots, e_{r-1}\}$ and each of $\{e_1, \dots, e_{r-1}\}$ appears in at most two such sets $E_G(C_i, V(G) - S - V(C_i))$, and so it yields

$$\sum_{i=1}^m e_G(C_i, V(G) - S - V(C_i)) \leq 2|\{e_1, e_2, \dots, e_{r-1}\}| = 2(r - 1).$$

Therefore

$$\begin{aligned} rm &\leq \sum_{i=1}^m (e_G(C_i, S) + e_G(C_i, V(G) - S - V(C_i))) \\ &\leq \sum_{x \in S} d_G(x) + \sum_{i=1}^m e_G(C_i, V(G) - S - V(C_i)) \\ &\leq r|S| + 2(r - 1). \end{aligned}$$

Hence $m \leq |S| + 2(r-1)/r < |S| + 2$, which implies $m \leq |S|$ by parity. That is, $o(H-S) \leq |S|$ or H has a 1-factor. $\square$

Note that from Theorem 1.4.3, we can give a short proof to Theorem 1.4.2 (a): for a given edge $e = xy$, let the set of other $r-1$ edges incident to the vertex x be X , then there exists a 1-factor F excluding X and thus F is a 1-factor containing e .

Also note that the above result holds for multigraphs as well.

This result is best possible in the sense that there exist infinitely many $(r-2)$ -edge-connected r -regular graphs of even order that have no 1-factor.

Claw-free Condition:

A graph containing no $K_{1,3}$ as an induced subgraph is said to be *claw-free*. Claw-free graphs have been a popular condition to study for many graphic parameters. An important reason is that under claw-free condition many graph theory problem will become significantly simpler. For instance, to determine a maximum independent set in arbitrary graphs is a *NP*-hard problem but becomes a *P*-problem if restricting to claw-free graphs.

Now we consider 1-factors in claw-free graphs.

Theorem 1.4.4 (Sumner [539])

- (a) *If G is a connected claw-free graph of even order, then has a 1-factor;*
- (b) *If a k -connected graph ($k \geq 2$) of even order has no induced subgraph isomorphic to $K_{1,k+1}$, then G has a 1-factor.*

Proof. (a) Use induction on $|G|$. Let $P = v_1, v_2, \dots, v_k$ be a longest path of G . Then $k \geq 3$. It follows $N_G(v_1) \subseteq V(P) - \{v_1\}$.

Claim. $G - \{v_1, v_2\}$ is connected.

If $d_G(v_2) = 2$, then $G - \{v_1, v_2\}$ is connected since $N_G(v_1) \subseteq V(P) - \{v_1\}$. If $d_G(v_2) \geq 3$. For every $x \in N_G(v_2) - V(P)$, xv_3 must be an edge of G since, otherwise v_1x or v_1v_3 is an edge of G as the vertices x, v_1, v_2, v_3 do not induce a claw. However, this implies that G contains a path longer than P , a contradiction.

Therefore $G - \{v_1, v_2\}$ is connected, and of course, is claw-free, and has even order. By the induction hypothesis, $G - \{v_1, v_2\}$ has a 1-factor and so does G .

(b) Suppose, to the contrary, that G has no 1-factor. By 1-Factor Theorem, then there exists a subset $S \subseteq V(G)$ such that $o(G-S) \geq |S| + 2$ by parity. Assume that S is a minimal such set and let $|S| = m$. Denote the odd components of $G-S$ by $O_1, O_2, \dots, O_t$. Thus $t \geq m+2$. Moreover, $|\{v \in S \mid v \text{ is adjacent to a vertex of } O_i\}| \geq k$, for each $i = 1, 2, \dots, t$, and so there are at least k edges from O_i to S having distinct end-vertices in S . Choosing such a collection of k edges for each i gives a collection

of $kt \geq k(m+2)$ edges and hence some vertex $x \in S$ is incident with at least $k+1$ of these edges. But then x is the center of an induced $K_{1,k+1}$, a contradiction. $\square$

The above theorem can be generalized in several ways. Here we mention one by Ryjáček [509] and others are due to Nebeský [445]. We say that a graph G is *almost claw-free* if the centers of induced claws in G are independent and their neighborhoods are 2-dominated. It can be showed that every almost claw-free graph is $K_{1,5}$ -free.

Theorem 1.4.5 (Ryjáček [509]) *Every connected almost claw-free graph of even order has a 1-factor.*

Given a graph G , if it does not have 1-factors, by 1-Factor Theorem, there exists a vertex set S_0 such that $o(G-S_0) > |S_0|$ and $o(G-S_0)-|S_0| = \max_{S \subseteq V(G)} o(G-S)-|S|$ (Note that this set is called a *barrier set*). Sumner [538] showed that there exists a barrier set S_0 , each vertex of S_0 is a claw center. In fact, a minimal barrier set is such a set. With this remark, Tutte's 1-Factor Theorem can be restated in the following form.

Theorem 1.4.6 (Sumner [538]) *A graph G has 1-factor if and only if for any set S consisting of claw centers,*

$$o(G-S) \leq |S|.$$

Neighborhood Condition:

Next we discuss the existence of 1-factors in terms of neighborhood condition.

Theorem 1.4.7 (Win [591]) *Let G be a graph of even order n . If for any two nonadjacent vertices x and y ,*

$$d_G(x) + d_G(y) \geq n + 1,$$

then G has a hamiltonian cycle disjoint from a 1-factor. In particular, G has three disjoint 1-factors.

Theorem 1.4.8 (Anderson [55]) *Let G be a graph of even order n . If, for any subset $S \subseteq V(G)$ with $|S| \leq \frac{3n}{4}$, we have $|N_G(S)| \geq \frac{4}{3}|S|$, then G has a 1-factor.*

Proof. Suppose, to the contrary, that G has no 1-factor. By 1-Factor Theorem, then there exists a vertex-subset $S_0 \subseteq V(G)$ such that $o(G-S_0) > |S_0|$.

Case 1. $|S_0| \geq \frac{n}{4}$.

Set $S = V(G) - S_0$. Then $|S| \leq \frac{3}{4}n$ and thus $|N_G(S)| \geq \frac{4}{3}|S|$. That is,

$$\frac{4}{3}(n - |S_0|) \leq |N_G(G - S_0)| \leq n - i(G - S_0),$$

i.e.,

$$i(G - S_0) \leq \frac{4}{3}|S_0| - \frac{n}{3}.$$

Thus

$$\begin{aligned} n &\geq |S_0| + i(G - S_0) + 3(o(G - S_0) - i(G - S_0)) \\ &> |S_0| + 3|S_0| - \frac{8}{3}|S_0| + \frac{2}{3}n. \end{aligned}$$

Hence $|S_0| < \frac{n}{4}$, a contradiction.

Case 2. $|S_0| < \frac{1}{4}n$.

Set $S = V(G) - S_0$. Then $|S| > \frac{3}{4}n$. Let S' be any subset of S with $|S'| = \frac{3}{4}n$. Then $|N_G(S')| \geq \frac{4}{3}|S'| = n$ and so $V(G) \subseteq N_G(S)$. Thus the vertices of S are adjacent to all the vertices of G . Consequently, there are no singleton components in $G - S_0$ and thus there are at least $|S_0| + 2$ odd components with at least 3 vertices in $G - S_0$. Consider all but one of these components. There are, say, h vertices in their union, where $h \geq 3|S_0| + 3$. But they can be adjacent to at most $h + |S_0| \leq h + \frac{1}{3}(h - 3) < \frac{4}{3}h$ vertices. This contradiction completes the proof. $\square$

Note that the value $\frac{4}{3}$ in the above theorem is best possible. To see this, consider the graph $G = (r + 2)K_3 \vee (rK_1)$. Let $S = V(rK_1)$. Then $o(G - S) = r + 2 > |S| = r$ and thus G has no 1-factor. Let $S' = V((r + 1)K_3)$. The $|N_G(S')| = 3(r + 1) + r$. Since $(4r + 3)/(3r + 3)$ can be made as close to $\frac{4}{3}$ as we like, it follows that the value $\frac{4}{3}$ is best possible.

Degree Sequences:

Let F be a 1-factor of a graph G . Then $G - F$ is a subgraph with the degree one less than that in G for each vertex, i.e., $d_{G-F}(x) = d_G(x) - 1$ for any $x \in V(G)$. This obvious fact turns out to be also sufficient for the existence of 1-factors in a graph. The next theorem gives a condition for the existence of 1-factors in terms of realization of degree sequences.

Theorem 1.4.9 (Kundu [332]; Lovász [410]) *Let G be a graph of even order with degree sequence $d_1, d_2, \dots, d_n$. If $d_1 - 1, d_2 - 1, \dots, d_n - 1$ is a degree sequence of some graph, then G has a 1-factor.*

Proof. Let $d_1, d_2, \dots, d_n$ be the degree sequence of G and $d_1 - 1, d_2 - 1, \dots, d_n - 1$ the degree sequence of another graph G' with $V = V(G) = V(G')$. Choose G and G' so that the symmetric difference of $E(G)$ and $E(G')$, $E(G) \triangle E(G')$, is minimal. To show that G has a 1-factor, we need only to show that $G' \subseteq G$.

Suppose that $G' \not\subseteq G$, i.e., $E(G') \not\subseteq E(G)$. Let x_0 be a vertex which is incident with the most number, say r , of edges in $E(G') - E(G)$. Since $d_G(x_0) = d_{G'}(x_0) + 1$, the definition of symmetric difference implies that there are $r + 1$ edges in $E(G) - E(G')$

incident with x_0 . Due to the choice of x_0 and $E(G') \not\subseteq E(G)$, there exists a vertex $z \in V$ such that $zx_0 \in E(G') - E(G)$ and then we pick up another vertex $w \in V$ with $zw \in E(G) - E(G')$. Note that such a w exists since $d_G(z) = d_{G'}(z) + 1$.

Claim 1. For any $y \in V - \{x_0, w\}$ with $x_0y \in E(G) - E(G')$, then $wy \in E(G) - E(G')$.

Otherwise, then $wy \notin E(G)$ or $wy \in E(G')$. If $wy \notin E(G)$, then we replace the edges x_0y and wz by the edges x_0z and wy in G . The resulting graph has the same degree sequence, but with smaller symmetric difference with respect to G' , a contradiction to the minimality. Similarly, if $wy \in E(G')$, then replacing the edges x_0z and wy by the edges x_0y and wz in G' yields a contradiction, again.

From Claim 1, for every edge $yx_0 \in E(G) - E(G')$, we have the edge $yw \in E(G) - E(G')$ ($y \neq x_0, w$). In addition, there is also the edge wz . Note that the edge x_0w may or may not be an edge of $E(G) - E(G')$, but it contributes the same number to $d_G(x_0)$ and $d_G(w)$ in any case. Thus, we see that the number of edges in $E(G) - E(G')$ incident with w is greater than the number of edges in $E(G) - E(G')$ incident with x_0 , a contradiction to the choice of x_0 . $\square$

The above theorem can also be rewritten as a sufficient and necessary condition for the realization of degree sequence in graphs with 1-factors.

Corollary 1.4.1 *Let $d_1, d_2, \dots, d_n$ be integers. Then $d_1, d_2, \dots, d_n$ is a degree sequence of a graph with a 1-factor if and only if n is even and both sequences $d_1, d_2, \dots, d_n$ and $d_1 - 1, d_2 - 1, \dots, d_n - 1$ are degree sequences of graphs.*

Induced Subgraph and 1-Factors:

The existence of 1-factors in subgraphs can also ensure the existence of 1-factors in the original graph. This fact is firstly proved by Sumner [539]. In fact, this idea turns out to be very fruitful and can be applied to study many properties related to matchings. We shall present several results about extendability in induced subgraphs in Section 6.2 later.

Theorem 1.4.10 (Sumner [539]) *If G is a connected graph of order $2n$ ($n \geq 2$) and k is an integer, $1 \leq k \leq n$, such that every connected induced subgraph of order $2k$ has a 1-factor, then G has a 1-factor.*

Proof. We use induction on n . It is easy to check the validity of the theorem for $n = 2, 3$. Suppose that it holds for some $n - 1$ ($n \geq 4$). Let G and k be given in the theorem. Clearly, we may assume that $k \leq n - 1$, since the case of $k = n$ is trivial.

It is easy to see that G does not contain two leaves adjacent to a same vertex z , otherwise these two leaves together with z can be extended to a connected induced

subgraph of order $2k$ which has no 1-factor.

Claim 1. There exist two adjacent vertices $x, y \in V(G)$ such that $G - \{x, y\}$ is connected.

Without loss of generality, assume that G is not complete and thus G has diameter $d \geq 2$. Let $a, y \in V(G)$ be vertices of G which are a distance d apart and let $P = a \dots xy$ be a path of length d joining a and y . Suppose that $G' = G - \{x, y\}$ is not a connected graph. Let C be the component of G' containing a . Since P is a diameter of G , every vertex in $G' - C$ is adjacent to x . Thus, if $G' - C$ contains a nontrivial component B , then for any $b, c \in B$ which are adjacent, every vertex in $G - \{b, c\}$ is joined to x by a path, and hence $G - \{b, c\}$ is connected. Similarly, if there exists a vertex $e \in G' - C$ which is adjacent to y , then $G - \{e, y\}$ is connected. Hence we may suppose that $G' - C$ contains only isolated vertices and each of these vertices is adjacent only to x . Thus, every vertex in $G' - C$ is a leaf of G adjacent to x . Therefore, since G has no two leaves adjacent to a same vertex, $G' - C$ must consist of a single isolated vertex f . If y were a leaf of G , then f and y would be leaves adjacent to a same vertex. Hence, we may assume that y is adjacent to some vertex of C , and so $G - \{x, y\}$ is connected.

From Claim 1, $G - \{x, y\}$ is connected and inductive assumption implies that $G - \{x, y\}$ has a 1-factor F . Hence, $F \cup xy$ constitutes a 1-factor of G . $\square$

Theorem 1.4.11 (Sumner [539]) *If G is a connected graph of even order and every induced $K_{1,3}$ contains a bridge whose deletion results in two components of even order, then G has a 1-factor.*

Proof. Use induction on $|G|$. It is easy to check the theorem holds for small orders. Assume the theorem holds for graphs with fewer vertices than $|G|$.

If G has no induced $K_{1,3}$, then G has a 1-factor by Theorem 1.4.4 (a). If G does contain an induced $K_{1,3}$, let $e = ab$ be an edge of this $K_{1,3}$ such that the deletion of e results in a graph having two even components A and B with $a \in A$ and $b \in B$. Suppose A contains an induced $K_{1,3}$. Let f be an edge of this subgraph such that $G - f$ contains two components of even order. Since both end-vertices of f lie in A , $A - f$ consists of two components C and D . We may assume that $a \in D$. Then C is one of the components of $G - f$ and hence has even order (as must D also). Thus A satisfies the condition of the theorem and so must have a 1-factor by induction hypothesis. Similarly, B contains a 1-factor which together with that in A produces a 1-factor in G . $\square$

Other Conditions:

Many other parameters, e.g., toughness, binding number, independent number, genus, etc., can be used as sufficient conditions for 1-factors. Below we include two such theorems as conclusion of this section.

Let G be a connected graph. The *toughness* of G was introduced by Chvátal [153] and defined as

$$t(G) = \min\left\{\frac{|S|}{\omega(G-S)} \mid S \subseteq V(G), \omega(G-S) \geq 2\right\},$$

if G is not a complete graph; $t(G) = \infty$ if G is a complete graph. If $t(G) \geq k$, then we call G *k-tough*.

Theorem 1.4.12 (Chvátal [153]) *Let G be a connected 1-tough graph of even order. Then G has a 1-factor.*

Proof. For any nonempty subset $S \subset V(G)$, if $\omega(G-S) \geq 2$, then

$$|S| \geq \frac{|S|}{t(G)} \geq \omega(G-S) \geq o(G-S),$$

or $o(G-S) \leq |S|$. If $\omega(G-S) = 1$, then $o(G-S) \leq \omega(G-S) = 1 \leq |S|$ since S is nonempty. When $S = \emptyset$, then $o(G-S) = o(G) = 0 = |S|$ as G is of even order. That is, $o(G-S) \leq |S|$ for any subset S and thus G has a 1-factor. $\square$

The last parameter to consider is genus. The *genus*, $\gamma^*(G)$, of a graph G is defined to be smallest genus (i.e., number of handles) of all surfaces on which G can be embedded.

Theorem 1.4.13 (Nishizeki [466]) *Let G be a n -connected graph of even order. If $\gamma^*(G) < n(n-2)/4$, then G has a 1-factor.*

1.5 Gallai-Edmonds Structure Theorem

In this section, we present a structural theorem about matchings in graphs, which is a refinement of Tutte's 1-Factor Theorem. It is also the foundation for Edmonds' famous polynomial algorithm of maximum matchings. This theorem is a powerful accomplishment and it gives much more deeper characterizations to matchings as a subgraph. However, it seems this result has not reached its full potential yet. It is our hope that more applications and appreciation of this theorem are gained in the future.

The structure developed in the proof of Tutte 1-Factor Theorem (i.e., Theorem 1.3.1) is very handy in many proofs of factor related theorems. Typically, in order to prove the existence of 1-factors, one uses a contra-positive argument by assuming the graph with the given conditions does not have 1-factors. Without loss of generality, we may choose a *maximal* such graph H , i.e., H has no 1-factor but if any edge is added, the resulting graph does. Such graphs possess many useful properties, we state it in the next theorem.

Theorem 1.5.1 (see [414]) *A graph H of even order is maximal non-1-factorable if and only if it has the following structure:*

G consists of vertex-disjoint complete graphs $S, G_1, G_2, \dots, G_k$ such that $k = |S| + 2$, where $G_1, \dots, G_k$ are of odd order and every vertex of G_i is connected to every vertex of S .

Proof. (outline of the proof)

Let G be a maximal non-1-factorable graph of even order. Define $S = \{x \mid d_G(x) = |V(G)| - 1\}$. We complete the proof by showing

- (1) $o(G - S) = |S| + 2$ (use parity and maximality);
- (2) there are no even component in $G - S$;
- (3) each component of $G - S$ is a complete graph.

□

Based on Tutte's 1-Factor Theorem, we can easily obtain formulas for the deficiency and the matching number of a graph.

Theorem 1.5.2 (Berge's Formula) *Let $\text{def}(G)$ denote the deficiency of a graph G . Then*

$$\text{def}(G) = \max_{S \subseteq V(G)} \{o(G - S) - |S|\}, \quad (1.6)$$

$$\mu(G) = \frac{1}{2} \left(\min_{S \subseteq V(G)} \{|V(G)| - o(G - S) + |S|\} \right). \quad (1.7)$$

Proof. Let $d = \text{def}(G)$. At first, we notice that a matching can cover at most $|V(G)| - d$ vertices.

On the other hand, by Tutte's 1-Factor Theorem, the graph $H = G \vee K_d$ has a 1-factor. Those edges of a 1-factor of H that belongs to G do not cover at most d vertices of G .

Since $\mu(G) = \frac{1}{2}(|V(G)| - \text{def}(G))$, (1.7) follows immediately. □

We defined a *barrier* as a set $S \subseteq V(G)$ such that $o(G - S) = |S| + \text{def}(G)$. So barriers are such sets which reach the maximum in Berge's Formula. If G has a 1-factor, then $\emptyset$ is a barrier.

From Tutte's 1-Factor Theorem, we see that a graph G of even order has no 1-factor if and only if there exists a subset S such that $G - S$ has at least $|S| + 2$ odd components. To further understand the structure of arbitrary graphs from the matching point of view, we refine Tutte's condition and introduce a canonical decomposition. This decomposition was studied by Gallai (1963) and Edmonds (1965), independently.

Figure 1.7: A barrier S , where $\text{def}(G) = 2$ and $\mu(G) = 14$

Recall that a graph G is *factor-critical* if $G - v$ has a 1-factor for every vertex v of G . It is easy to see that if G is factor-critical, then G is of odd order, connected and not a bipartite graph.

For a set $S \subseteq V(G)$, an *odd* (resp. *even*) S -*component* is a connected component of the graph $G - S$ of odd (resp. even) cardinality. The total number of vertices in those S -components which have no 1-factor is denoted by $\text{Def}(S)$, while the quantity $\text{def}(S) = o(G - S) - |S|$ is called the *deficiency* of S . Directly from the definitions, we have the following two observations.

Observations:

- (1) Every matching in G leaves at least $\text{def}(S)$ vertices unsaturated;
- (2) if a matching M leaves exactly $\text{def}(S)$ vertices unsaturated, then M contains a 1-factor of each even S -component, a near-perfect matching of each odd S -component, and matches all vertices of S to vertices in distinct odd S -components.

For a vertex set S and a matching M , if M has exactly $\text{def}(S)$ vertices unsaturated (i.e., (2) holds), then S is called a *Tutte-Berge set*, while M is maximum by (1). The Berge's Formula (1.6) asserts that a Tutte-Berge set S always exists.

If $|S| + o(G - S) \geq 2$, we denote by $\langle G, S \rangle$ the bipartite graph obtained from G by deleting all the even S -components, contracting each odd S -component to a single vertex, and deleting the edges spanned by S . For a bipartite graph $H = (A, B)$, the set A satisfies Hall's condition (resp., with *positive surplus*) if $|N_H(X)| \geq |X|$ (resp., $|N_H(X)| > |X|$) for every non-empty subset X of A . Therefore, Hall's Theorem asserts that H has a matching saturated every vertex of A if and only if A satisfies Hall's condition. Finally, we call S a *Gallai-Edmonds set* with respect to G , or a *G-E set* in short, if S satisfies

- (a) all even S -components (if any) have a 1-factor;
- (b) all odd S -components (if any) are factor-critical;
- (c) if $S \neq \emptyset$, then S satisfies Hall's condition with positive surplus in $\langle G, S \rangle$.

Now we are ready to present and prove the famous Gallai-Edmonds Structure Theorem, which is a deep theorem for matchings. It provides a clear structure for graphs without 1-factors and it is the foundation of Edmonds' polynomial algorithm for finding a maximum matching in an arbitrary graph. It deserves more attention than its current status in authors' opinion. The proof used here is given by Kotlov [323] and it derives Gallai-Edmonds Structure Theorem from Hall's Theorem.

Theorem 1.5.3 (Gallai-Edmonds Structure Theorem) *Let G be a graph. Then*

- (i) *there exists a Gallai-Edmonds set S ;*
- (ii) *S is a Tutte-Berge set. That is, every maximum matching of G contains a near-perfect matching of each odd S -component, a 1-factor of each even S -component, and matches all vertices of S to vertices in distinct S -components;*
- (iii) *the underlying vertex set of all odd S -components is the set $D(G)$ of the vertices left unsaturated by at least one maximum matching of G , while S is the neighborhood of $D(G)$. In particular, G has a unique Gallai-Edmonds set.*

Proof. (i) Among all vertex subsets of G with maximum deficiency, let S have minimum $\text{Def}(S)$ and $|S| = s$. We use induction on $|G|$ to show that S is a G-E set.

Suppose that C is an even S -component without 1-factors. For any fix $v \in V(C)$ and let $S' = S \cup \{v\}$. Then $\text{def}(S') \geq \text{def}(S)$ while $\text{Def}(S') < \text{Def}(S)$. This contradiction shows that S satisfies (a).

Suppose that v is a vertex in an odd S -component C such that $C - v$ has no 1-factor. By induction hypothesis, the graph $H = C - v$ has a G-E set T . In particular, $\text{def}_H(T) \geq 2$. Set $S' = S \cup T \cup \{v\}$. Then $\text{def}(S') = o(G - S') - |S'| = (o(G - S) + o(H - T) - 1) - (s + |T| + 1) = \text{def}(S) + \text{def}_H(T) - 2 \geq \text{def}(S)$ while $\text{Def}(S') < \text{Def}(S)$, a contradiction. Thus, S satisfies (b).

Suppose that X is a smallest non-empty subset of S violating Hall's condition with positive surplus in $\langle G, S \rangle$. Set $S' = S - X$. If X consists of a single vertex with no neighbors in $\langle G, S \rangle$, then $\text{def}(S') > \text{def}(S)$, which is a contradiction. Otherwise, X satisfies Hall's condition in $\langle G, S \rangle$. By (b) and Hall's Theorem, X is contained in an even S' -component with a 1-factor (this even component is spanned by X and the $|X|$ odd S -components neighboring with X in G). Consequently, $\text{def}(S') = \text{def}(S)$ and $\text{Def}(S') < \text{Def}(S)$. Thus, S satisfies (c).

Let now S be a G-E set, and v a fixed vertex in an odd S -component C . By (c) and Hall's Theorem, S can be matched to vertices in s distinct odd S -components different from C . By (a) and (b), this matching can be extended by (near-)perfect matchings of the S -components to obtain a matching M avoiding v . Since M leaves $\text{def}(S)$ vertices unsaturated, it is maximum, and (ii) and (iii) follow immediately. $\square$

From Theorem 1.5.3, the G-E set plays an important role in the structure of a graph. We can actually determine this G-E set explicitly as follow: for a graph G , denote by $D(G)$ the set of all vertices in G which are *not* saturated by *at least* one maximum matching of G . Let $A(G)$ be the neighbor set of $D(G)$, i.e., the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$. Finally let $C(G) = V(G) - D(G) - A(G)$. Clearly, this partition is well-defined for every graph and does not rely on the choices of maximum matchings. In fact, the set $A(G)$ is our G-E set. The partition $(D(G), A(G), C(G))$ is called a *canonical decomposition*.

This partition of $V(G)$ is also refinement of Tutte's 1-factor Theorem, $A(G)$ can be thought as a Tutte-set S (which is a barrier), $D(G)$ is the union of the odd components of $G - S$ and $C(G)$ is the union of the even components of $G - S$. As an example, Figure 1.8 shows the canonical decomposition $(D(G), A(G), C(G))$.

With the canonical decomposition $(D(G), A(G), C(G))$, we can restate Theorem 1.5.3 in the following form, which is the usual formation in most literatures.

Theorem 1.5.4 (Gallai-Edmonds Structure Theorem) *Let G be a graph and let $D(G)$, $C(G)$ and $A(G)$ be the vertex-partition defined above. Then*

- (i) *the components of the subgraph induced by $D(G)$ are factor-critical;*
- (ii) *the subgraph induced by $C(G)$ has a 1-factor;*
- (iii) *if M is any maximum matching of G , it contains a near-perfect matching of each component of $D(G)$, a 1-factor of each component of $C(G)$ and matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$;*
- (iv) *the bipartite graph obtained from G by deleting the vertices of $C(G)$ and the edges spanned by $A(G)$ and by contracting each component of $D(G)$ to a single vertex has positive surplus (as viewed from $A(G)$);*

Figure 1.8: The canonical decomposition $(D(G), A(G), C(G))$

(v) the size of maximum matching is $\frac{1}{2}(|V(G)| - \omega(D(G)) + |A(G)|)$.

Note: For a bipartite graph $G = (X, Y)$, the *surplus* of a set $S \subseteq X$ is defined by $|N(S)| - |S|$. The *surplus* of G is the minimum surplus of all non-empty subset of X .

Clearly, Theorem 1.5.4 provides more information than Tutte's 1-Factor Theorem. In fact, the latter can be easily derived from former. To see this, note that since $o(G - S) \leq |S|$ holds for all $S \subseteq V(G)$, in particular $o(G - A(G)) \leq |A(G)|$ and then substituting this into the formula in (v), we have $\frac{1}{2}(|V(G)| - \omega(D(G)) + |A(G)|) \geq \frac{1}{2}|V(G)|$. Thus G has a 1-factor. Similarly, let $S = A(G)$ and (v) yields equality of Berge's Formula.

In fact, the structure provided by Gallai-Edmonds Theorem has many other important consequences as well. Many of the results in this chapter are consequences of this powerful theorem.

Remark 1.5.1. (1) From the part (v), it is immediate that the deficiency of G , $\text{def}(G) = \omega(D(G)) - |A(G)|$; that is, it is the number of components of $D(G)$ left unmatched into $A(G)$ by an arbitrary maximum matching of G .

(2) If G has no 1-factor, then every edge incident with a vertex of $D(G)$ lies in some maximum matching of G . To see this, let $u \in D(G)$ and uv be any edge incident to u . By the definition of $D(G)$, there is a maximum matching missing u . Hence by the maximality of M , there is an edge vw in M which covers v . But then $M - vw \cup uv$ is the desired matching.

It follows from (iv) that no edge induced by $A(G)$ or connecting $A(G)$ to $C(G)$ belongs to any maximum matching. The edges in the subgraph induced by $C(G)$ show a much more complicated behavior in this respect and this problem will be discussed later in Chapter 2.

- (3) If G itself is a factor-critical graph, then we have a degenerate case of the canonical decomposition; namely, $D(G) = V(G)$, so $A(G) = C(G) = \emptyset$.
- (4) If G has a 1-factor, we have another degenerate case; namely, $D(G) = \emptyset$, so $A(G) = \emptyset$ and hence $C(G) = V(G)$.
- (5) If G is a bipartite graph with bipartition (X, Y) and if it has positive surplus (from X), then $A(G) = X$ and $D(G) = Y$. To see this, note that $\mu(G) = |X|$ and Hall's Theorem implies that $G - v$ has a maximum matching of X into $Y - v$ for every $v \in Y$. Hence $Y = D(G)$ and $X = A(G)$.

Thus the Gallai-Edmonds canonical decomposition is trivial in three cases: factor-critical graphs, graphs with 1-factors and bipartite graphs with positive surplus. On the other hand, it tells us that every graph can be built up from graphs of these three types.

Besides the properties mentioned in Theorem 1.5.4 and Remark 1.5.1, the sets $D(G)$, $C(G)$ and $A(G)$ have other interesting features as well. The next assertion was used as a lemma to prove Theorems 1.5.3 and 1.5.4 in several literatures. Even though we do not need it for the proof of Theorems 1.5.3, the assertion has its own merit to be mentioned.

Theorem 1.5.5 (Stability Lemma, see [414]) *Let G be a graph and let $D(G)$, $C(G)$ and $A(G)$ be defined above. Then*

- (i) *for any $u \in A(G)$, then $A(G-u) = A(G)-u$, $C(G-u) = C(G)$, and $D(G-u) = D(G)$;*
- (ii) *for any $u \in C(G)$, then $A(G-u) \supseteq A(G)$, $C(G-u) \subseteq C(G)-u$, and $D(G-u) \supseteq D(G)$;*
- (iii) *for any $u \in D(G)$, then $A(G-u) \subseteq A(G)$, $C(G-u) \supseteq C(G)$, and $D(G-u) \subseteq D(G)-u$.*

Proof. We only give the detailed proof of part (i) here. For parts (ii) and (iii), it can be proved similarly (see [414]).

(i) From the definitions of $D(G)$, $C(G)$ and $A(G)$, it is clear that we need only to show $D(G-u) = D(G)$.

Let M be a maximum matching of G and thus $\mu(G) = |M|$. Since $u \notin D(G)$, then M covers u . Hence $\mu(G-u) = \mu(G) - 1$. Moreover, since M is a maximum matching of G , $M - u$ is a maximum matching of $G - u$.

To see $D(G) \subseteq D(G - u)$, choose any $v \in D(G)$, let M_v be a maximum matching of G which misses v . Then $M_v - u$ is a maximum matching of $G - u$ and moreover, $M_v - u$ misses v too, so $v \in D(G - u)$. Thus $D(G) \subseteq D(G - u)$.

To see $D(G - u) \subseteq D(G)$, choose any $v \in D(G - u)$. Then there is a maximum matching M' of $G - u$ which misses v . Let w be any vertex in $D(G)$ adjacent to u in G and let M be a maximum matching of G misses w . If M misses v , then $v \in D(G)$ follows, so we assume that M covers v . Consider $M \cup M'$. By definition, M' avoids v . Thus the component of $M \cup M'$ covering v must be a path P starting at v with an edge of $M - M'$.

Suppose P ends with an edge of M' . Then $M \triangle P$ is a new matching M'' in G missing v . Moreover, $|M''| = |M|$, so M'' is a maximum matching and $v \in D(G)$. Otherwise, P ends with an edge of M . Let $M''' = M' \triangle P$. Then M''' is a matching and $|M'''| > |M|$. So $M''' \not\subseteq E(G - u)$ by the maximality of M' . Thus P must end at u . But now $M \triangle E(P \cup uw)$ is a maximum matching of G avoiding v and thus $v \in D(G)$. $\square$

For bipartite graphs, we have the following structural theorem.

Theorem 1.5.6 (see [414]) *Let $G = (U_1, U_2)$ be a connected bipartite graph, and let $C_1 = C(G) \cap U_1$ and $C_2 = C(G) \cap U_2$. Then the following statements hold*

- (i) $G[D(G)]$ is an independent set G ;
- (ii) $G[C(G)]$ has a 1-factor, and $|C_1| = |C_2|$;
- (iii) every maximum matching M in G consists of a 1-factor of $G[C(G)]$ and a matching in $G[A(G) \cup D(G)]$ saturating $A(G)$;
- (iv) both $A(G) \cap C_1$ and $A(G) \cap C_2$ are minimum vertex covers of G ;
- (v) both $D(G) \cap C_1$ and $D(G) \cap C_2$ are maximum independent subsets of G .

As an application of Gallai-Edmonds Structure Theorem, we prove the following result regarding the connectivity of allowed edges. More applications of Theorem 1.5.4 can be found in Section 5.1.

Theorem 1.5.7 (see [414]) *Let G be a connected graph with 1-factors. If $C(G - x) = \emptyset$ for all $x \in V(G)$, then all allowed edges form a connected spanning subgraph of G .*

Proof. Suppose, by the contrary, that there exists a partition of $V(G)$, say (V_1, V_2) where $V_1 \neq \emptyset \neq V_2$, such that all edges of $E(V_1, V_2)$ are not allowed, i.e., for any 1-factor M , we have $M \cap E(V_1, V_2) = \emptyset$. Thus M must match V_i onto itself and so $|V_i|$ is even ($i = 1, 2$).

Claim 1. There exists a subset S such that $|S| \geq 2$ and $o(G - S) = |S|$.

Let $v_1v_2 \in E(V_1, V_2)$, where $v_1 \in V_1$ and $v_2 \in V_2$. Then v_1v_2 is not allowed, that is, $G' = G - \{v_1, v_2\}$ has no 1-factor. From 1-Factor Theorem, there exists a subset $S' \subseteq V(G')$ such that $o(G' - S') > |S'|$ or $o(G' - S') \geq |S'| + 2$ by parity. Let $S = S' \cup \{v_1, v_2\}$. Then $o(G - S) = o(G' - S') \geq |S'| + 2 = |S|$. On the other hand, since G has 1-factors, we have $o(G - X) \leq |X|$ for any subset $X \subseteq V(G)$. Thus $o(G - S) = |S|$ or S is a barrier.

Without loss of generality, we assume that S is a *maximal* barrier.

Claim 2. Every edge $e = xy$ with $x \in S$ and $y \notin S$ is allowed in G .

Consider the graph $G - x$. Since $\text{def}(G) = 0$, so $\text{def}(G - x) = 1$. Because of $o(G - S) = |S|$, $x \in S$ and $\text{def}(G - x) = 1$, every maximum matching of $G - x$ saturates every vertex of $S - x$. By the definition of $D(G - x)$, so $S - x \subseteq A(G - x) \cup C(G - x) = A(G - x)$ as $C(G - x) = \emptyset$. On the other hand, by Stability Lemma (i.e., Theorem 1.5.5), $A(G - x) = A(G) - x$ and thus $A(G - x) \cup x = A(G)$ is a barrier in G . By the maximality of S , $A(G - x) \cup x \subseteq S$. Thus

$$S = A(G - x) \cup x. \quad (1.8)$$

Since $y \notin S$, by (1.8), $y \in D(G - x) \cup C(G - x) = D(G - x)$ as $C(G - x) = \emptyset$ again. So there exists a maximum matching M_x of $G - x$ missing y but saturating all other vertices of $G - x$. But then $M_x \cup \{xy\}$ is a 1-factor of G , i.e., xy is an allowed edge of G .

Claim 3. $G - S$ has no odd component joined to both $S \cup V_1$ and $S \cap V_2$.

Suppose, to the contrary, that H is such a component. Let uv be an edge with $u \in V(H)$ and $v \in S \cap V_1$. By Claim 2, there exists a 1-factor F of G containing uv . Thus $u \in V_1$ by the definition of V_1 .

Since $|V(G)|$ is even, we have $D(G - v) \neq \emptyset$ but $C(G - v) = \emptyset$. Let e_v be an edge of F which is incident to v . Then $F - e_v$ is a maximum matching of $G - v$. Moreover, $F - e_v$ saturates all vertices of $A(G - v)$ and has at most one edge joining each odd component of $D(G - v)$ (of which H is one) to $A(G - v)$. Recall $S = A(G - v) \cup v$, upon reinstalling v and e_v we have exactly one edge of F joining S to each component of $G - S$ and in particular, exactly one edge of F joining H to S . So $E(H) \cap F$ is a near-perfect matching of H . Thus $E(H - V_2) \cap F$ is a near-perfect matching of $H - V_2$, and so $|V(H) \cap V_1|$ is odd. Similarly, $|V(H) \cap V_2|$ is odd and hence $|V(H)|$ is even, a contradiction.

By Claims 2 and 3, all odd components of $G - S$ are joined to $S \cap V_1$ or $S \cap V_2$, but not both. For $i = 1, 2$, let a_i be the number of odd components of $G - S$ joined to $S \cap V_i$. Then $a_1 + a_2 = o(G - S) = |S|$. Without loss of generality, assume $a_1 \leq |S \cap V_1|$. Let $x \in S \cap V_2$. Then, from (1.8), $A(G - v)$ has a subset of $k = |S \cap V_1|$ vertices joined to at most k odd components of $(G - x) - A(G - x)$, which contradicts to the part (iv) of Gallai-Edmonds Structure Theorem.

The proof is complete. $\square$

Kelmans [314] studied the *induced* star packing problem as a natural generalization of the matching problem and obtained a Gallai-Edmonds type of structure theorem for induced star packing and also included Tutte's 1-Factor Theorem, Berge's Formula, the matching matroid theorem as special cases.

Pulleyblank [503] extend Theorem 1.5.4 to fractional matchings. We will provide more details in Section 8.1.

1.6 Number of 1-factors

Let $\Phi(G)$ denote the number of 1-factors in a graph G . One of focuses in the study of matching theory is to determine the function $\Phi(G)$. However, it seems very unlikely that a formula or an efficient algorithm for computing $\Phi(G)$ will ever be found. In particular, Valiant (1979) proved that the problem of determining $\Phi(G)$ is NP-hard, even when G is bipartite. This left very little room for finding the exact value of $\Phi(G)$. Naturally, the next move is to find a lower bound for $\Phi(G)$.

Let's first present a few classical results in this direction.

Theorem 1.6.1 (Hall (1948)) *Let $G = (X, Y)$ be a bipartite graph with a 1-factor and $d(x) \geq k$ for every $x \in X$. Then $\Phi(G) \geq k!$.*

Theorem 1.6.2 (Beineke and Plummer (1967)) *Let G be a 2-connected graph with a 1-factor. Then $\Phi(G) \geq 2$.*

Theorem 1.6.3 (Zaks (1970)) *Let G be a k -connected graph with a 1-factor ($k \geq 2$). Then $\Phi(G) \geq k!!$. In particular, $\Phi(K_n) = (n-1)!! = (n-1)(n-3) \dots 1$. The bounds are sharp when k is odd.*

Theorem 1.6.4 *Let G be a connected graph of order n with a unique 1-factor. Then*

- (a) G has a bridge belonging to the 1-factor;
- (b) G contains a vertex of degree $\leq \lfloor \log_2(n+1) \rfloor$; and
- (c) $|E(G)| \leq \frac{1}{4}n^2$.

Note that, in the above theorem, (a) is from Kotzig (1959); (b) is due to Lovász (1972) and (c) is from Heteyi (unpublished manuscript, see [406]).

Theorem 1.6.5 (Lovász [406]) *Let G be a k -connected graph with a 1-factor and G is not bicritical. Then $\Phi(G) \geq k!$*

An immediate consequence of Theorem 1.6.5 is that a k -connected graph G with 1-factors has $\Phi(G) \geq k!$ or G is bicritical. Bicritical graphs are a family of important graphs, and it plays a unreplaceable role in the study of matchings, for example, it can be used to develop a decomposition theory for elementary graphs (ref. Section 5.2)

and it can also be used to determine the dimension of 1-factors lattice (ref. Section 5.4).

From the definition of bicriticality, it seems that this family of graphs are very rich in 1-factors. However, bicritical graphs is somewhat counterintuitive. In fact, bicritical graphs contain much less 1-factors than other graphs and is more difficult to bound $\Phi(G)$ than most other graphs, as shown in the next theorem. This theorem has a weaker condition, 2-connected and $\delta(G) \geq k$, than Theorem 1.6.5.

Theorem 1.6.6 (Mader (1976)) *Let G be a 2-connected graph with $\delta(G) \geq k$ and have a 1-factor. Then either*

- (a) *there are at least $k!$ 1-factors in G (i.e., $\Phi(G) \geq k!$), or*
- (b) *G is bicritical. In this case, $\Phi(G) \geq |V(G)|/2 + 1$.*

The investigation of lower bound for general graphs is still ongoing. On the other hand, the lower bounds for some special families of graphs are also researched. For example, Lovász and Plummer [415] obtained the best possible lower bound $\Phi(H)$ for Halin graphs H (see Theorem ?? for the details).

It has been shown that one can bound $\Phi(G)$ below by a certain matrix function called Pfaffian. In the case when G is planar, the Pfaffian matrices can be used to exactly compute $\Phi(G)$ in polynomial time. (The definition of Pfaffian orientation can be found in Section 5.4)

Now we turn to the problem of finding a lower bound for the number of near-perfect matchings in a factor-critical graphs. The following result was obtained by Pulleyblank (1973) and he used linear programming technique to prove it. Here we present a short proof from [414].

Theorem 1.6.7 (Pulleyblank [502]) *Every 2-connected factor-critical graph G contains at least $|E(G)|$ near-perfect matchings.*

Proof. Since G is a 2-connected factor-critical graph, from Theorem ??, so G can be expressed as $G = P_0 + P_1 + P_2 + \cdots + P_m$, where P_0 is an odd cycle and each P_r ($1 \leq r \leq m$) is an open path of odd length joining two different vertices of $P_0 + P_1 + \cdots + P_{r-1}$. Denote that $G_r = P_0 + P_1 + \cdots + P_r$. Then G_r is factor-critical.

We use induction on m . If $m = 0$, the result clearly holds. Suppose the result holds for $r \geq 0$. We consider $G_{r+1} = G_r + P_{r+1}$, which is a 2-connected factor-critical graph. Let $P_{r+1} = xu_1 \dots u_{l-1}y$, where $l = 1$ or $l \geq 3$.

If F is any near-perfect matching of G_r , then $F' = F \cup \{u_1u_2, \dots, u_{l-2}u_{l-1}\}$ is a near-perfect matching of G_{r+1} . By induction hypothesis, there are $|E(G_r)| = |E(G_{r+1})| - l$ near-perfect matchings in G_{r+1} .

Since G_r is factor-critical, for each u_i ($1 \leq i \leq l-1$) we can easily find a 1-factor F_{u_i} of $G_{r+1} - u_i$. So we obtain $l-1$ more near-perfect matchings of G_{r+1} . Note that

each near-perfect matching F_{u_i} saturates only one of x and y with an edge of P_{r+1} . Next we construct the last near-perfect matching of G_{r+1} , which saturates both of x and y with the edges of P_{r+1} . Let F_x be a 1-factor of $G_r - x$. Then F_x must cover y with an edge $yz \in E(G_r)$. Therefore $F'' = (F_x - \{yz\}) \cup \{xu_1, u_2u_3, \dots, u_{l-1}y\}$ is a 1-factor of $G_{r+1} - z$ and thus the desired near-perfect matching of G_{r+1} .

We complete the induction. $\square$

Now we divert our attention from determining the value of $\Phi(G)$ to study of the structure of graphs which are sensitive to $\Phi(G)$. The most results in this area are due to Lovász [406]. Reader may choose to read this part after Chapter 5 (in particular, Section 5.1) since some terms and results are given in details there.

A graph G with a 1-factor is Φ -maximal if $\Phi(G \cup e) > \Phi(G)$ for any edge $e \notin E(G)$, i.e., the graph $G \cup e$ has more 1-factors than G . Note that a 1-extendable graph is Φ -maximal if and only if it is bicritical. Every graph with 1-factors is contained in a Φ -maximal graph because we can add edges to a graph as long as no new 1-factor are generated. Many important graph-theoretic properties are preserved by this procedure.

A subset $S \subseteq V(G)$ is called *extreme* if $\text{def}(G - S) = \text{def}(G) + |S|$. From the definition of extreme sets, it is easy exercises to see the following properties of extreme sets.

Lemma 1.6.1 *Let G be a graph. Then*

- (a) *every subset of an extreme set is extreme;*
- (b) *if S is an extreme set and $S \subseteq T$, then $T - S$ is an extreme set too.*

Below, we characterize Φ -maximal graphs.

Theorem 1.6.8 *Let G be any graph with 1-factors. Then G is Φ -maximal if and only every extreme set in G induces a complete graph, all edges of which are forbidden in G .*

Proof. Assume that G is Φ -maximal. Let X be an extreme set in G and $\{x, y\} \subseteq X$. By Lemma 1.6.1, $\{x, y\}$ is an extreme set of G .

Suppose x and y are not adjacent in G . Then $\text{def}((G \cup xy) - x - y) = \text{def}(G - x - y) = \text{def}(G) + 2$ since $\{x, y\}$ is extreme in G . But $\text{def}(G \cup xy) = \text{def}(G) = 0$, so we have $\text{def}((G \cup xy) - x - y) = \text{def}(G \cup xy) + 2$ and thus $\{x, y\}$ is extreme in $G \cup xy$. Therefore, xy is forbidden in $G \cup xy$ and so G is not Φ -maximal, a contradiction. Then we must have $xy \in E(G)$ and thus xy is forbidden in G .

Conversely, suppose G is not Φ -maximal and thus there are two vertices $x, y \in V(G)$ with $xy \notin E(G)$, such that G and $G \cup xy$ have the same 1-factors. Thus xy is forbidden in $G \cup xy$ and $\{x, y\}$ is extreme in $G \cup xy$. So $\text{def}(G - x - y) = \text{def}((G \cup xy) - x - y) = \text{def}(G \cup xy) + 2 = 0 + 2 = \text{def}(G) + 2$ and thus $\{x, y\}$ is

extreme in G . But this contradicts the hypothesis that xy must be a forbidden edge of G . $\square$

In addition, if a Φ -maximal graph is also elementary (i.e., allowed edges form a connected subgraph), then Theorem 1.6.8 yields the following.

Theorem 1.6.9 *If G is a Φ -maximal elementary graph, then the forbidden edges of G constitute vertex-disjoint complete subgraphs induced by the classes of $\mathcal{P}(G)$.*

The next result shows us that the Construction of general elementary graphs discussed in Section 5.2 is “more unique” when graphs are Φ -maximal.

Theorem 1.6.10 *Let G be a Φ -maximal elementary graph and let G^* be the graph obtained by contracting each class S of $\mathcal{P}(G)$ to a vertex t_S . Then*

- (a) *the block of G^* (i.e., the maximal 2-connected subgraph of G^*) are bicritical;*
- (b) *the number of blocks of G^* containing t_S is exactly $|S|$;*
- (c) *if we build G by iterating the construction of Theorem ??, then the initial bi-critical graphs must be precisely the blocks of G^* , and*
- (d) *the elementary bipartite graphs used in the last steps of the Construction of G are uniquely determined in the sense that they must be of the form G'_S , for some $S \in \mathcal{P}(G)$.*

Now we proceed to the main result of this section – Cathedral Theorem – for all Φ -maximal non-elementary graphs. First, we introduce Cathedral Construction, which was studied by Kotzig (1960):

Cathedral Construction. Let G_0 be any Φ -maximal elementary graph. To each class $S \in \mathcal{P}(G_0)$ assign an (already constructed) Φ -maximal graph G_S or the empty set. For each $S \in \mathcal{P}(G_0)$ join every vertex of S to every vertex of G_S . If G_S is not empty, we call the subgraph G_S the *tower* over S and call S the *foundation* of that tower. (Note that a tower and its foundation are vertex-disjoint.)

Now we can state Cathedral Theorem, but the proof is long and involved and we omit it.

Theorem 1.6.11 (Cathedral Theorem)

- (a) *Every graph G , built up by iterating Cathedral Construction using smaller Φ -maximal graphs, is itself Φ -maximal;*
- (b) *the allowed edges of G are precisely those edges which are allowed in one of the elementary graphs used in one of the steps.*

Figure 1.9: Construction of a Φ -maximal elementary graph

- (c) *Conversely, if G is any Φ -maximal graph, it can be built up using Cathedral Construction starting with a Φ -maximal elementary graph G_0 and a collection of $|\mathcal{P}(G_0)|$ smaller Φ -maximal graphs (some perhaps empty) already constructed. The graph G_0 may be uniquely described as the subgraph of G induced by those vertices of G which, for each $x \in V(G)$, does not lie in $C(G - x)$.*

An illustration of Cathedral structure is shown in Figure 1.10.

Determining $\Phi(G)$ for a graph G is an extremely difficult task, even for a special class of graphs, e.g., bipartite graph. One of important tools in studying $\Phi(G)$ of *bipartite graphs* is to use the permanent of incident matrix of G . There is a direct correspondence between binary $n \times n$ matrices and balanced bipartite graphs of order $2n$. So to compute the exact value of $\Phi(G)$ is equivalent to find the permanent

Figure 1.10: A Cathedral Construction

of corresponding matrices. Even though to determine the permanent of matrices is again a *NP*-complete problem, however the equivalence between the permanent and $\Phi(G)$ leads to the discovery of a good non-trivial lower bound for $\Phi(G)$ when G is bipartite and regular. The lower bound and upper bound are obtained by Schrijver and Brègman, respectively.

Theorem 1.6.12 (Schrijver (1998); Brègman (1973)) *Let G be a k -regular bipartite graph on $2n$ vertices. Then*

$$\left(\frac{(k-1)^{k-1}}{k^{k-2}} \right)^n \leq \Phi(G) \leq (k!)^{n/k}$$

As a conclusion of this section, as well as the chapter, we pose a conjecture on $\Phi(G)$ by Lovász and Plummer (1989).

Conjecture 1.6.1 (Lovász and Plummer [418]) *Let G be a 2-edge-connected cubic graph G . Then there exists a constant $c > 1$ such that $\Phi(G) \geq c^n$.*

There are some partial results known for this conjecture. For example, Voorhoeve (1979) showed that if G is a cubic bipartite graph on $2n$ vertices, then $\Phi(G) \geq (4/3)^n$.

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