The place of super edge-magic labelings among other classes of labelings

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Dedicated to Professor Gary Chartrand

Abstract

A \((p, q)\)-graph \(G\) is edge-magic if there exists a bijective function \(f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}\) such that \(f(u) + f(v) + f(uv) = k\) is a constant, called the valence of \(f\), for any edge \(uv\) of \(G\). Moreover, \(G\) is said to be super edge-magic if \(f(V(G)) = \{1, 2, \ldots, p\}\). In this paper, we present some necessary conditions for a graph to be super edge-magic. By means of these, we study the super edge-magic properties of certain classes of graphs. We also exhibit the relationships between super edge-magic labelings and other well-studied classes of labelings. In particular, we prove that every super edge-magic \((p, q)\)-graph is harmonious and sequential (for a tree or \(q \geq p\)) as well as it is cordial, and sometimes graceful. Finally, we provide a closed formula for the number of super edge-magic graphs. \(\copyright\) 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently, a conference paper by Ringel [13] has sparked renewed interest in the study of edge-magic labelings of graphs, which originally were introduced and studied by Kotzig and Rosa [10,11], who called them magic valuations. The authors were particularly intrigued by Ringel’s remark (during his oral presentation of the paper) to the effect that he knew of no relationships between this type of labeling and other well-known classes of labelings. Later, Enomoto et al. [3] restricted the notion of edge-magic labelings to obtain the definition of super edge-magic labeling of a graph. This new
The definition has led the authors to find new relationships between super edge-magic labelings and other well-studied classes of labelings: graceful, harmonious, sequential and cordial labelings. Of particular interest among these is our result that every super edge-magic \((p,q)\)-graph is harmonious (if it is a tree or \(q \geq p\)). This coupled with the fact that several classes of graphs (connected and disconnected) have recently been shown to be super edge-magic [3,4] opens a new avenue of assault to the problem of finding classes of harmonious graphs. Furthermore, the authors have found that trees that admit \(x\)-valuations are super edge-magic and hence are harmonious. Now, since \(x\)-valuations of trees are well studied, we have then further evidence of the validity of the conjecture by Graham and Sloane [8] that all trees are harmonious.

Another aspect of super edge-magic labelings that caught the authors’ attention was that there were relatively few techniques and necessary conditions to show whether a graph is super edge-magic. To this end, we have accumulated some new necessary conditions and have used them in conjunction with the existing ones to analyze the super edge-magic properties of certain classes of graphs. Of particular interest are some of the results we have obtained about graphs that are edge-magic but not super edge-magic in spite of satisfying the necessary conditions.

The authors were able to find a closed formula for the number of super edge-magic graphs.

Finally, we point out what we believe is the major reason for interest in super edge-magic graphs. After sometime working on these problems, it appears to us that the definition of super edge-magic labeling is restrictive enough so that one has more conditions to lay siege to labeling problems, yet it is general enough to allow for a wealth of non-trivial results.

Now, we provide the definition for the two key concepts to be discussed in this paper.

An edge-magic labeling of a \((p,q)\)-graph \(G\) is a bijective function

\[ f: V(G) \cup E(G) \rightarrow \{1,2,\ldots,p+q\} \]

such that \(f(u) + f(v) + f(uv) = k\) is a constant for any edge \(uv\) of \(G\). In such a case, \(G\) is said to be edge-magic and \(k\) is called the valence of \(f\). Moreover, \(f\) is a super edge-magic labeling of \(G\) if \(f(V(G)) = \{1,2,\ldots,p\} \) and \(G\) is said to be super edge-magic.

The reader is directed to Chartrand and Lesniak [2] or Hartsfield and Ringel [9] for all additional terminology not provided in this paper.

2. Necessary conditions

In this section, we present several necessary conditions for a graph to be super edge-magic.

The following lemma provides a necessary and sufficient condition for a graph to be super edge-magic, which is most of the time more useful than the definition itself.
Lemma 1. A \((p,q)\)-graph \(G\) is super edge-magic if and only if there exists a bijective function \(f: V(G) \rightarrow \{1, 2, \ldots, p\}\) such that the set 
\[ S = \{ f(u) + f(v) : uv \in E(G) \} \]
consists of \(q\) consecutive integers. In such a case, \(f\) extends to a super edge-magic labeling of \(G\) with valence \(k = p + q + s\), where \(s = \min(S)\) and 
\[ S = \{ k - (p + 1), k - (p + 2), \ldots, k - (p + q) \}. \]

Proof. First, assume that such a function \(f\) exists and let \(xy \in E(G)\) so that \(f(x) + f(y) = \min(S) = s\). Then \(f\) extends to the domain \(V(G) \cup E(G)\) in the following manner. Let \(f(uv) = p + q + s - f(u) - f(v)\) for any edge \(uv\) of \(G\). Then \(f(E(G)) = \{ p + 1, p + 2, \ldots, p + q \}\).

Conversely, if \(G\) is super edge-magic with a super edge-magic labeling \(f\) of valence \(k\), then 
\[ S = \{ k - f(uv) : uv \in E(G) \} \]
\[ = \{ k - (p + 1), k - (p + 2), \ldots, k - (p + q) \}. \]

In light of this result, it suffices to exhibit the vertex labeling of a super edge-magic graph. However, we will also provide the valences to increase the clarity of our results.

The next corollary will prove later to be very useful. Furthermore, we would like to point out that it has provided us with a good starting point for computer searches of super edge-magic labelings of some graphs.

Corollary 2. Let \(G\) be a super edge-magic \((p,q)\)-graph and \(f\) be a super edge-magic labeling of \(G\). Then 
\[ \sum_{v \in V(G)} f(v) \deg v = qs + \binom{q}{2}, \]
where \(s\) is defined as in the previous lemma. In particular, 
\[ 2 \sum_{v \in V(G)} f(v) \deg v \equiv 0 \pmod{q}. \]

The following corollary excludes certain graphs from the class of super edge-magic graphs whose components are eulerian.

Corollary 3. Let \(G\) be a \((p,q)\)-graph, where every vertex of \(G\) is even and \(q \equiv 2 \pmod{4}\), then \(G\) is not super edge-magic.

The next result is particularly useful in showing that a regular graph is not super edge-magic.
Lemma 4. If $G$ is an $r$-regular super edge-magic $(p,q)$-graph, where $r > 0$, then $q$ is odd and the valence of any super edge-magic labeling of $G$ is $\frac{1}{2}(4p + q + 3)$.

Proof. The valence of any super edge-magic labeling of $G$ is

$$\frac{1}{q} \left\{ r \sum_{i=1}^{p} i + \sum_{i=p+1}^{p+q} i \right\} = \frac{1}{2}(4p + q + 3),$$

which implies that $q$ is odd. $\Box$

The following useful lemma was found by Enomoto et al. [3].

Lemma 5. If a $(p,q)$-graph is super edge-magic, then $q \leq 2p - 3$.

Notice that if $q = 2p - 3$, then the vertices labeled with the following pairs of integers have to be adjacent, $(1,2)$, $(1,3)$, $(p, p - 2)$ and $(p, p - 1)$ since there is a unique way of expressing $3$, $4$, $2p - 2$ and $2p - 1$ as sums of two distinct elements in the set \{1,2,\ldots, p\}.

As a corollary to Lemma 5, we get the following result.

Corollary 6. Every super edge-magic $(p,q)$-graph contains at least two vertices of degree less than 4.

Proof. Assume, to the contrary, that $p - 1$ vertices of $G$ have degree at least 4. Then, by the first theorem of graph theory and the previous lemma

$$4p - 4 = \sum_{i=1}^{p-1} 4 \leq \sum_{v \in V(G)} \deg v = 2q \leq 2(2p - 3) = 4p - 6$$

which is a contradiction. $\Box$

This implies that the minimum degree is at most 3 for every super edge-magic graph. Thus, in light of Whitney’s inequality (see [2, p. 152, Theorem 5.1] for example), the inequality $\kappa(G) \leq \kappa_1(G) \leq 3$ holds for every super edge-magic graph $G$, where $\kappa(G)$ and $\kappa_1(G)$ denote the connectivity and edge-connectivity of $G$, respectively.

It is well known that the $n$-dimensional cube $Q_n$ is $n$-regular. Hence, it is not super edge-magic for $n \geq 4$ by the previous corollary and for $n = 2$ and $n = 3$, we obtain a regular graph of even size, which is impossible by Lemma 4. Therefore, $Q_n$ is super edge-magic if and only if $n = 1$. Also, the toroidal mesh $C_m \times C_n$ is excluded by the previous corollary for every pair of integers $m \geq 3$ and $n \geq 3$.

We end this section with the following result.

Lemma 7. Let $G$ be a super edge-magic graph of size $q$ and $f$ be a super edge-magic labeling of $G$. Then there are exactly $\left\lfloor \frac{q}{2} \right\rfloor$ or $\left\lceil \frac{q}{2} \right\rceil$ edges between $V_e$ and $V_o$, where $V_e = \{v \in V(G) : f(v) \text{ is even}\}$.
and
\[ V_o = \{ v \in V(G) : f(v) \text{ is odd} \}. \]

**Proof.** Since \( f \) is a super edge-magic labeling of \( G \), it follows that the set:
\[ S = \{ f(u) + f(v) : uv \in E(G) \} \]
consists of \( q \) consecutive integers. Then \([\frac{q}{2}] \) or \([\frac{q}{2}] \) of the elements in \( S \) are odd and each of these has to be the result of adding the label of an element in \( V_e \) to the label of an element in \( V_o \). \( \square \)

3. The super edge-magic properties of certain graphs

With the results in the previous section in hand, we are ready to study the super edge-magic properties of certain graphs. Of particular interest are those classes presented here that satisfy the above necessary conditions and are however not super edge-magic.

The following theorem is interesting because it analyzes some \((p,q)\)-graphs for which \( q = 2p - 3 \).

**Theorem 8.** The fan \( f_n \cong P_n + K_1 \) is super edge-magic if and only if \( 1 \leq n \leq 6 \).

**Proof.** First, we show that \( f_n \) is super edge-magic for \( 1 \leq n \leq 6 \). The graphs \( f_1 \cong K_2 \) and \( f_2 \cong K_3 \) are both trivially super edge-magic. For \( n = 3, 4, 5 \) and 6, label \( K_1 \) with 4 and \( P_n \) with \( 3 - 1 - 2, 5 - 3 - 1 - 2, 6 - 5 - 3 - 1 - 2 \) and \( 6 - 7 - 5 - 3 - 1 - 2 \), respectively.

For the converse, assume, to the contrary, that \( f_n \) is super edge-magic with a super edge-magic labeling \( g \) for every integer \( n \geq 7 \). Then define \( p = n + 1, q = 2n - 1 \), and \( V(f_n) = \{ v_i : g(v_i) = i \} \). Now, since \( f_n \) is super edge-magic, it follows from Lemma 1 that \( S = \{ g(u) + g(v) : uv \in E(f_n) \} \) is a set of \( q = 2p - 3 \) consecutive integers, implying that \( S = \{3, 4, \ldots, 2p - 1\} \). Since \( n \geq 7 \), the vertices \( v_1, v_2, v_3, v_4, v_{p-3}, v_{p-2}, v_{p-1} \) and \( v_p \) are all distinct.

Observe next that each of 3, 4, 2\( p - 2 \) and 2\( p - 1 \) can be expressed uniquely as sums of two distinct elements from the set \( L = \{1, 2, \ldots, p\} \), namely, \( 3 = 1 + 2, 4 = 1 + 3, 2p - 2 = p + (p - 2) \) and \( 2p - 1 = p + (p - 1) \). Therefore, \( \{v_1v_2, v_1v_3, v_{p-2}v_{p-1}v_{p-1}v_p\} \subseteq E(f_n) \).

Also, notice that the integers 5 and 2\( p - 3 \) can be expressed each in exactly two ways as sums of distinct elements of \( L \), namely, \( 5 = 1 + 4 = 2 + 3 \) and \( 2p - 3 = p + (p - 3) = (p - 2) + (p - 1) \). Thus, there are four mutually exclusive possibilities: either \( \{v_1v_4, v_pv_{p-3}\}, \{v_1v_4, v_{p-1}v_{p-2}\}, \{v_2v_3, v_pv_{p-3}\} \) or \( \{v_2v_3, v_{p-1}v_{p-2}\} \) is a subset of \( E(f_n) \).

Finally, by adding any of these four pairs of edges to the four edges that are necessarily in \( E(f_n) \), we obtain a forbidden subgraph of \( f_n \), namely, either \( 2K_{1,3}, K_{1,3} \cup K_3 \) or \( 2K_3 \). \( \square \)

The fan is however always edge-magic as is shown in the next theorem.
Theorem 9. The fan $f_n$ is edge-magic for every positive integer $n$.

Proof. Let $f_n$ be the fan with
\[ V(f_n) = \{u\} \cup \{v_i : 1 \leq i \leq n\} \]
and
\[ E(f_n) = \{uvi : 1 \leq i \leq n\} \cup \{vivi+1 : 1 \leq i \leq n-1\}. \]

Now, construct the function $f : V(f_n) \cup E(f_n) \rightarrow \{1, 2, \ldots, 3n\}$ as follows:
\[
f(x) = \begin{cases} 
1 & \text{if } x = u, \\
\frac{1-5(-1)^{v_i+6i}}{4} & \text{if } x = v_i \text{ and } 1 \leq i \leq n, \\
\frac{12n+7+5(-1)^{v_i+6i}}{4} & \text{if } x = uv_i \text{ and } 1 \leq i \leq n, \\
3n - 3i + 1 & \text{if } x = vivi+1 \text{ and } 1 \leq i \leq n-1. 
\end{cases}
\]

Notice that $f(x) + f(y) + f(xy) = 3n + 3$ for any edge $xy$ of $f_n$. Also, observe that
\[
\begin{align*}
\{f(v_{2i+1}) : 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\} &= \{3i : 1 \leq i \leq \left\lfloor \frac{n+1}{2} \right\rfloor\}, \\
\{f(uv_{2i}) : 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\} &= \{3i : \left\lfloor \frac{n+1}{2} \right\rfloor + 1 \leq i \leq n\}, \\
\{f(v_iv_{i+1}) : 1 \leq i \leq n-1\} &= \{3i+1 : 1 \leq i \leq n-1\}, \\
\{f(v_{2i}) : 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\} &= \{3i+2 : 0 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor\}, \\
\{f(uv_{2i+1}) : 0 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor\} &= \{3i+2 : \left\lfloor \frac{n-2}{2} \right\rfloor + 1 \leq i \leq n-1\},
\end{align*}
\]
and $f(u) = 1$, so all the integers 1 through $3n$ are used exactly once. Therefore, $f$ is an edge-magic labeling of $f_n$ with valence $3n + 3$. \qed

The authors have been informed through personal communication with Enomoto [16] that K. Yokomura has also proven independently the following two results about ladders and generalized prisms.

Theorem 10. The ladder $L_n \cong P_n \times P_2$ is super edge-magic, where $n$ is odd.

Proof. Let $L_n$ be the ladder with
\[ V(L_n) = \{u_i, v_i : 1 \leq i \leq n\} \]
and
\[ E(L_n) = \{u_iu_{i+1}, v_iv_{i+1}, uv_j : 1 \leq i \leq n-1, 1 \leq j \leq n\}. \]
Now, consider the following function:

\[ f : V(L_n) \rightarrow \{1, 2, \ldots, 2n\}, \]

where

\[
\begin{align*}
  f(x) = \begin{cases} 
    \frac{i+1}{2} & \text{if } x = u_i, \, i \text{ odd and } 1 \leq i \leq n, \\
    \frac{n+i+1}{2} & \text{if } x = u_i, \, i \text{ even and } 1 \leq i \leq n, \\
    \frac{3n+i}{2} & \text{if } x = v_i, \, i \text{ odd and } 1 \leq i \leq n, \\
    \frac{2n+i}{2} & \text{if } x = v_i, \, i \text{ even and } 1 \leq i \leq n.
  \end{cases}
\end{align*}
\]

We conclude that \( f \) extends to a super edge-magic labeling of \( L_n \) with valence \((11n+1)/2\).

The converse of the previous theorem is not true. Although the graph \( L_2 \cong C_4 \) is not super edge-magic by Lemma 4, we have found super edge-magic labelings for \( n = 4 \) and \( n = 6 \). In the case \( n = 4 \), label one \( P_4 \) with \( 1 - 5 - 4 - 3 \) and the other one with \( 7 - 6 - 8 - 2 \); and, for \( n = 6 \), label one \( P_6 \) with \( 1 - 5 - 7 - 11 - 8 - 2 \) and the other one with \( 6 - 3 - 10 - 4 - 12 - 9 \). We suspect that a pattern might be found for larger even values of \( n \).

The next theorem shows that the generalized prism \( C_m \times P_n \) is sometimes super edge-magic. Notice that the construction of it given in the proof is intended to make the vertex labeling easy to describe.

**Theorem 11.** The generalized prism \( C_m \times P_n \) is super edge-magic if \( m \) is odd and \( n \geq 2 \).

**Proof.** The generalized prism \( G \cong C_m \times P_n \) can be defined as follows:

\[ V(G) = \{v_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \]

and

\[ E(G) = \{v_{i,j}v_{i+1,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{v_{i,j}v_{i,j+1} : 1 \leq i \leq m, 1 \leq j \leq n - 1\}, \]

where \( i \) is taken modulo \( m \) (replacing 0 by \( m \)).

Now, consider the following function \( f : V(G) \rightarrow \{1, 2, \ldots, mn\} \), where

\[
\begin{align*}
  f(v_{i,j}) = \begin{cases} 
    \frac{i+1}{2} & \text{if } 1 \leq i \leq m \text{ is odd and } j = 1, \\
    \frac{i+m+1}{2} & \text{if } 1 \leq i \leq m \text{ is even and } j = 1, \\
    \frac{i+m(2j-2)}{2} & \text{if } 1 \leq i \leq m \text{ is even and } 2 \leq j \leq n, \\
    \frac{i+m(2j-1)}{2} & \text{if } 1 \leq i \leq m \text{ is odd and } 2 \leq j \leq n.
  \end{cases}
\end{align*}
\]

We conclude that \( f \) extends to a super edge-magic labeling of \( G \) whose valence is \((6mn - m + 3)/2\). \( \square \)
It is important to notice that the converse of the previous result is an immediate
case where \( n = 2 \) and \( m \) is odd is interesting
because \( r = 0, 1, 2 \) or \( 3 \) for \( r \)-regular super edge-magic graphs by Lemma 4.
The next result presents strong necessary conditions for the book \( B_n \cong K_{1,n} \times K_2 \) to
be super edge-magic.

**Theorem 12.** If the book \( B_n \) is super edge-magic with a super edge-magic labeling \( f \)
such that
\[
s = \min \{ f(x) + f(y) : xy \in E(G) \}
\]
then the following conditions are satisfied:

1. If \( n \) is odd, then \( n \equiv 5 \pmod{8} \) and
   \[
s \in \left\{ \frac{n + 27}{8}, \frac{3n + 25}{8}, \frac{5n + 23}{8}, \frac{7n + 21}{8}, \frac{9n + 19}{8} \right\}
   \]
   unless \( n = 5 \), in which case, \( s \) can also be \( 3 \);
2. If \( n \) is even, then \( s = \frac{n}{2} + 3 \) unless \( n = 2 \), in which case, \( s \) can also be \( 3 \).

**Proof.** The book \( B_n \) has order \( p = 2n + 2 \) and size \( 3n + 1 \). Now, if \( x \) and \( y \) represent
the labels of the two vertices of degree \( n + 1 \) of \( B_n \), then
\[
2 \sum_{i=1}^{2n+2} i + (x + y)(n - 1) = (3n + 1)s + \frac{3n(3n + 1)}{2}
\]
by Corollary 2, so
\[
x + y = \frac{n^2 + 6sn - 17n + 2s - 12}{2n - 2}
\]
however, \( x + y \leq p + (p - 1) = 4n + 3 \). Consequently,
\[
3 \leq s \leq \frac{7n + 19}{6} + \frac{8}{27n + 9} \leq \frac{7n + 7}{3}
\]
since \( n \geq 1 \).

If \( n \) is even, then \( n = 2k \) for some integer \( k \), so
\[
x + y = k + 3s - 8 + \frac{4s - 14}{2k - 1}
\]
by (1) and hence \( 2k - 1 \) divides \( 2s - 7 \) for \( k \geq 2 \), that is, there exists an integer \( m \)
such that
\[
m(2k - 1) + 7 = s.
\]
Then, from (2), we obtain \( -1 \leq m \leq 2 \), implying that \( m = 1 \) since \( s \) is an integer and
\( k \geq 2 \). Hence, \( s = \frac{4}{2} + 3 \). For \( n = 2 \), notice that \( s = 3 \) or \( s = 4 \) by (2).
Some super edge-magic labelings of $B_n$

<table>
<thead>
<tr>
<th>$n$</th>
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<th>Labeling</th>
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<tbody>
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<td>2</td>
<td>4</td>
<td>(1; 3, 5), (6; 2, 4)</td>
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<td>4</td>
<td>(1; 3, 4, 5, 8, 9), (6; 12, 10, 7, 11, 2)</td>
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<td>5</td>
<td>7</td>
<td>(7; 1, 2, 3, 6, 11), (12; 10, 5, 9, 8, 4)</td>
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<tr>
<td>12</td>
<td>9</td>
<td>(8; 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13), (19; 15, 20, 21, 22, 24, 25, 16, 23, 18, 14, 26, 17)</td>
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</table>

For the cases where $n$ is odd, if $n \equiv 3 \pmod{4}$, then every vertex of $B_n$ is even and $q \equiv 2 \pmod{4}$, so $B_n$ is not super edge-magic by Corollary 3. On the other hand, if $n \equiv 1 \pmod{4}$, then $n = 4k + 1$ for some integer $k$ and

$$2(x + y) = 4k + 6s - 15 + \frac{2s - 7}{k}$$

by (3), which means that $k$ divides $2s - 7$.

Now, if $n = 8k + 1$ for some integer $k$, then $2k$ divides $2s - 7$, which is not possible. Therefore, when $n$ is odd, there exists an integer $k$ such that $n = 8k + 5$, so $2k + 1$ divides $2s - 7$ and hence there exists an integer $m$ such that

$$m(2k + 1) - 1 = 2s.$$ 

Then, from (2), we obtain $-1 \leq m \leq 9$; however, $m \in \{-1, 1, 3, 5, 7, 9\}$ since $s$ is an integer. Therefore,

$$s \in \left\{ \frac{-n + 29}{8}, \frac{n + 27}{8}, \frac{3n + 25}{8}, \frac{5n + 23}{8}, \frac{7n + 21}{8}, \frac{9n + 19}{8} \right\}.$$ 

Finally, notice that $s = (-n + 29)/8$ only when $n = 5$, which completes the proof.

An exhaustive computer search of the cases for which $2 \leq n \leq 5$ shows that the previous theorem can be strengthened for those values of $n$. First, $s$ can never be 3 when $n = 2$. Second, no super edge-magic labeling of $B_n$ exists for $n = 4$. Third, $s$ can take only the values 4 through 7 for $n = 5$. Now, Table 1 contains super edge-magic labelings of $B_n$ for all the possible cases up to 12. The vertex labelings are presented as $(n + 1)$-tuples, one for each star, the first element of each tuple is the label of the central vertex of the corresponding copy of $K_{1,n}$ and the vertex receiving the $i$th label in the first tuple is adjacent to its counterpart in the second one.

The above theorem, remark and Table 1 lead us to the following conjecture.

**Conjecture 13.** For every integer $n \geq 5$, the book $B_n$ is super edge-magic if and only if $n$ is even or $n \equiv 5 \pmod{8}$.
Although books are sometimes not super edge-magic, they are always edge-magic as the following theorem demonstrates.

**Theorem 14.** The book $B_n$ is edge-magic for any positive integer $n$.

**Proof.** Let $B_n$ be the book defined as follows:

$$V(B_n) = \{u, v\} \cup \{u_i, v_i: 1 \leq i \leq n\}$$

and

$$E(B_n) = \{uv\} \cup \{uu_i, vv_i, u_iv_i: 1 \leq i \leq n\}.$$

Then consider the function

$$f : V(B_n) \cup E(B_n) \rightarrow \{1, 2, \ldots, 5n + 3\},$$

where

$$f(x) = \begin{cases} 
1 & \text{if } x = u, \\
5n + 3 & \text{if } x = v, \\
2n + 2 & \text{if } x = uv, \\
2n + i + 2 & \text{if } x = u_i \text{ and } 1 \leq i \leq n, \\
2n - 2i + 2 & \text{if } x = v_i \text{ and } 1 \leq i \leq n, \\
5n - i + 3 & \text{if } x = uu_i \text{ and } 1 \leq i \leq n, \\
3n + i + 2 & \text{if } x = u_iv_i \text{ and } 1 \leq i \leq n, \\
2i + 1 & \text{if } x = vv_i \text{ and } 1 \leq i \leq n. 
\end{cases}$$

Finally, observe that $f$ is an edge-magic labeling of $B_n$ having valence $7n + 6$. 

4. Relationships with other labelings

This section places super edge-magic labelings in their proper place among other classes of labelings that have previously been well studied. The order in which we present these relationships is the one that we feel is most conducive to a coherent and brief presentation (as opposed to one that lists each kind of labeling by its relative importance). Thus, we start defining sequential labelings.

The definition of sequential labelings was introduced by Grace [7] and is inspired by harmonious labelings (which we will discuss shortly). A sequential labeling of a $(p, q)$-graph $G$ is an injective function $f : V(G) \rightarrow \{0, 1, \ldots, q - 1\}$ (with the label $q$ allowed if $G$ is a tree) such that the induced edge labeling given by $f(uv) = f(u) + f(v)$ has the property that

$$\{f(uv): uv \in E(G)\} = \{m, m + 1, m + 2, \ldots, m + q - 1\}$$

for some integer $m$. Moreover, $G$ is said to be sequential if such a labeling exists.

With this definition in hand, we now present the following result.
Theorem 15. If a \((p,q)\)-graph \(G\) that is a tree or where \(q \geq p\) is super edge-magic, then \(G\) is sequential.

Proof. Let \(f\) be a super edge-magic labeling of \(G\) with valence \(k\), then

\[\{f(u) + f(v) : uv \in E(G)\} = \{k - (p + 1), k - (p + 2), \ldots, k - (p + q)\}\]

by Lemma 1.

Now, define \(g : V(G) \rightarrow \{0,1,\ldots,p-1\}\) to be the injective function such that \(g(v) = f(v) - 1\) for each vertex \(v\) of \(G\). Hence,

\[\{g(u) + g(v) : uv \in E(G)\} = \{m, m+1, \ldots, m+q-1\},\]

where \(m = k - (p + q + 2)\), which implies that \(g\) is a sequential labeling of \(G\). □

Harmonious labelings have been defined and studied by Graham and Sloane [8] as part of their study of additive bases and are applicable to error-correcting codes. A harmonious labeling of a \((p,q)\)-graph \(G\) with \(q \geq p\) is an injective function \(f : V(G) \rightarrow \{0,1,\ldots,q-1\}\) satisfying the condition that induced edge labeling given by \(f(uv) \equiv f(u) + f(v) \pmod{q}\) for any edge \(uv\) of \(G\) is also an injective function. Furthermore, \(G\) is said to be harmonious if such a labeling exists. This definition extends to trees (for which \(q = p - 1\)) if at most one vertex label is allowed to be repeated.

Theorem 15 together with the fact that Grace [7] showed that sequential \((p,q)\)-graphs with \(q \geq p\) are harmonious yield the following result.

Theorem 16. If a \((p,q)\)-graph \(G\) with \(q \geq p\) is super edge-magic, then \(G\) is harmonious.

This theorem extends easily to trees and thus we obtain the next theorem.

Theorem 17. If a tree \(T\) of order \(p\) is super edge-magic, then \(T\) is harmonious.

Proof. Recall that \(q(T) = p - 1\) and then reduce the edge labels modulo \(p - 1\). □

This result implies that the conjecture by Enomoto et al. [3] that all trees are super edge-magic is at least as hard as the conjecture by Graham and Sloane that all trees are harmonious!

The most famous graph labeling problem that has been studied is that of finding graceful labelings of graphs, which were defined by Rosa [15]. These arose naturally out of the study of graph decompositions and the subsequent Ringel–Kotzig conjecture that all trees are graceful [12].

Let \(G\) be a \((p,q)\)-graph and \(f : V(G) \cup E(G) \rightarrow \{0,1,\ldots,q\}\) such that \(f(uv) = |f(u) - f(v)|\) for any edge \(uv\) of \(G\) and \(f|_{V(G)}\) and \(f|_{E(G)}\) are injective. Then \(f\) is a graceful labeling of \(G\) and it is called a graceful graph. Also, as a result of Rosa’s interest on graph decompositions, he defined what he called an \(\alpha\)-valuation of a graph.
A graceful labeling $f$ of a $(p,q)$-graph $G$ is said to be an $\alpha$-valuation of $G$ if there exists an integer $k$ with $0 \leq k < q$, called the characteristic of $f$, such that $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ for every edge $uv$ of $G$.

The next two theorems establish the relationships between super edge-magic labelings and $\alpha$-valuations.

**Theorem 18.** Suppose that $G$ is a super edge-magic bipartite $(p, p - 1)$-graph with partite sets $V_1$ and $V_2$, where $p_1 = |V_1|$ and $p_2 = |V_2|$, and let $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, 2p - 1\}$ be a super edge-magic labeling of $G$ such that $f(V_1) = \{1, 2, \ldots, p_1\}$. Then $G$ has an $\alpha$-valuation.

**Proof.** Consider a $(p, p - 1)$-graph $G$ and a super edge-magic labeling $f$ of $G$ such that both satisfy the hypothesis of the theorem. Furthermore, select the vertices of $G$ so that $V(G) = \{v \in V(G) : f(v) = 1\}$. Then

$$f(V_1) = \{1, 2, \ldots, p_1\} \quad \text{and} \quad f(V_2) = \{p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\}.$$

Now, let $g : V(G) \cup E(G) \rightarrow \{0, 1, \ldots, p - 1\}$ be the labeling such that

$$g(v) = \begin{cases} f(v) - 1 & \text{if } v \in V_1, \\ 2p_1 + p_2 - f(v) & \text{if } v \in V_2. \end{cases}$$

We next prove that $g$ is an $\alpha$-valuation of $G$ with characteristic $p_1 - 1$. First, observe that

$$g(V_1) = \{0, 1, \ldots, p_1 - 1\} \quad \text{and} \quad g(V_2) = \{p_1, p_1 + 1, \ldots, p_1 + p_2 - 1\}.$$

Also, if $u \in V_2$ and $v \in V_1$, then

$$|g(u) - g(v)| = g(u) - g(v) = 2p_1 + p_2 + 1 - (f(u) + f(v)).$$

Hence, $1 \leq |g(u) - g(v)| \leq p - 1$, since

$$p_1 + 2 \leq f(u) + f(v) \leq 2p_1 + p_2.$$

Finally, since $u \in V_2$ and $v \in V_1$ are arbitrary vertices of $G$, it suffices to observe that $\{f(u) + f(v) : uv \in E(G)\}$ is a set of $p - 1$ consecutive integers by Lemma 1, which implies that $g(E(G)) = \{1, 2, \ldots, p - 1\}$. \[\square\]

We comment here that Rosa [15] has shown that all graphs that admit $\alpha$-valuations are bipartite. Therefore, we have the converse of Theorem 18, which we state without proof.

**Theorem 19.** Let $G$ be a bipartite $(p, p - 1)$-graph with an $\alpha$-valuation $f$ such that there exists partite sets $V_1$ and $V_2$, where $p_1 = |V_1|$, $p_2 = |V_2|$ and $f(V_1) = \{0, 1, \ldots, p_1 - 1\}$, then $G$ is super edge-magic.

This theorem is important due to the following corollary.
Corollary 20. If $T$ is a tree having an $x$-valuation, then $T$ is super edge-magic.

A number of techniques to construct trees from smaller ones with $x$-valuations have been shown to yield $x$-valuations in the resulting trees. The reader is referred to the survey paper by Gallian [5] for references to these methods.

Cahit [1] defined cordial labelings of graphs as a way of stating a weaker condition that would reflect the spirit of both graceful and harmonious labelings. A cordial labeling of $G$ is a function $f: V(G) \rightarrow \mathbb{Z}_2$ with an induced edge labeling $f(\overline{uv}) \equiv f(u) - f(v) \pmod{2}$ such that if $v_f(i)$ and $e_f(i)$ are the number of vertices $v$ and edges $e$ satisfying that $f(v) = i$ and $f(e) = i$ for all $i \in \mathbb{Z}_2$, respectively, then $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus, a graph that admits a cordial labeling is said to be cordial.

With this definition, we are able to show the next relationship between labelings.

Theorem 21. If a graph $G$ is super edge-magic, then $G$ is cordial.

Proof. Let $G$ be super edge-magic with a super edge-magic labeling $f$. Then consider the function $g: V(G) \cup E(G) \rightarrow \mathbb{Z}_2$ such that $g(v) \equiv f(v) \pmod{2}$ for every vertex $v$ of $G$ and $g(\overline{uv}) \equiv g(u) - g(v) \pmod{2}$ for any edge $uv$ of $G$. Notice that $g(\overline{uv}) = g(u) - g(v) = g(u) + g(v) \equiv f(u) + f(v) \pmod{2}$.

Also, since $f(V(G))$ and $\{f(u) + f(v) : uv \in E(G)\}$ are sets of consecutive integers by definition of super edge-magic graph and Lemma 1, respectively, it follows that $|v_g(0) - v_g(1)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$. \qed

5. New edge-magic labelings from old

Kotzig and Rosa [10] defined the complementary labeling of an edge-magic labeling so that if $f$ is an edge-magic labeling of a $(p,q)$-graph $G$, then the complementary labeling to $f$ is the labeling $\tilde{f}$ of $G$ such that $\tilde{f}(x) = p + q + 1 - f(x)$ for every $x \in V(G) \cup E(G)$.

Notice that $g = \tilde{f}$ is an edge-magic labeling of $G$ and $\tilde{g} = f$.

The definition of complementary labeling inspires the following theorem.

Theorem 22. Let $T$ be an edge-magic tree of order $p$ with an edge-magic labeling $f$ whose valence is $k$ such that $f(v)$ is odd for any vertex $v$ of $T$. Then the bijective function $g: V(T) \cup E(T) \rightarrow \{1,2,\ldots,2p-1\}$ defined as

$$g(x) = \begin{cases} \frac{f(x)+1}{2} & \text{if } x \in V(T), \\ \frac{f(x)}{2} + p & \text{if } x \in E(T). \end{cases}$$

is a super edge-magic labeling. Furthermore, given a super edge-magic labeling of a tree, a labeling can be obtained with all vertices receiving an odd label by reversing the above process.
Proof. Notice first that if \( u \) and \( v \) are distinct vertices of \( T \), then \( g(u) \neq g(v) \). In addition, if \( e_1 \) and \( e_2 \) are different edges of \( T \), then \( g(e_1) \neq g(e_2) \). Next,
\[
1 \leq g(u) \leq p < g(e) \leq 2p - 1
\]
for every vertex \( u \) and every edge \( e \) of \( G \).

Finally, observe that
\[
g(u) + g(v) + g(uv) = \frac{f(u) + f(v) + f(uv) + 2}{2} + p = \frac{k}{2} + p + 1
\]
is an integer constant for each edge \( uv \) of \( G \) since \( k \) is even. \( \square \)

The authors would like to point out that when trying to find super edge-magic labelings of trees, the previous theorem has been very useful since often times we have been more successful in finding an edge-magic labeling of a tree with all vertices labeled with odd integers than directly providing a super edge-magic one!

We remark that Theorem 22 can also be extended to \((p,q)\)-graphs for which \( p = q \).

6. Counting

A well-known result by Gilbert [6] states that almost all graphs are connected, which implies that for almost all \((p,q)\)-graphs satisfy that \( q \geq p \). This combined with Graham and Sloane’s [8] result that almost all graphs are not harmonious and Theorem 16 leads us to the following theorem.

Theorem 23. Almost all graphs are not super edge-magic.

We are able to provide the following closed formula for the number of super edge-magic graphs.

Theorem 24. The number of distinct super edge-magic labelings of \((p,q)\)-graphs is
\[
\sum_{i=3}^{2p-q} \prod_{j=i}^{i+q-1} a(f),
\]
where
\[
a(f) = \begin{cases} \left\lfloor \frac{j-1}{2} \right\rfloor & \text{if } 3 \leq j \leq p + 1, \\ \left\lceil \frac{2p-j+1}{2} \right\rceil & \text{if } p + 2 \leq j \leq 2p - 1. \end{cases}
\]

Proof. Consider the complete graph \( K_p \) with \( V(K_p) = \{v_i : 1 \leq i \leq p\} \) and the labeling
\[
f : V(G) \cup E(G) \rightarrow \left\{ 1, 2, \ldots, p + \frac{p(p - 1)}{2} \right\}
\]
such that \( f(v_i) = i \) for every integer \( i \) with \( 1 \leq i \leq p \) and \( f(uv) = f(u) + f(v) \) for any edge \( uv \) of \( G \).
Let $A_j = \{uv \in E(G) : f(uv) = j\}$ and $a(j) = |A_j|$ for every integer $j$ with $3 \leq j \leq 2p-1$. Then, by Lemma 1, a vertex labeling $f$ of a $(p,q)$-graph $G$ with $f(V(G))$ extends to a super edge-magic labeling if $\{f(u) + f(v) : uv \in E(G)\}$ is a set of $q$ consecutive integers. Thus, a super edge-magic $(p,q)$-graph $G$ can be constructed from the labeling $f$ of $K_p$ by taking

$$V(G) = V(K_p) \quad \text{and} \quad E(G) = \{e_j \in A_j : i \leq j \leq i + q - 1\}$$

for some fixed integer $i$ with $3 \leq i \leq 2p - q$. Then a super edge-magic labeling of $G$ is obtained by restricting $f$ to $V(G) \cup E(G)$. Notice that $E(G)$ can be selected in $\prod_{j=i}^{i+q-1} a(j)$ ways.

Finally, if we take all possible integer values of $i$ such that $3 \leq i \leq 2p - q$, then the result follows immediately. \(\square\)

The following Table 2 shows the number of super edge-magic labelings of $(p,q)$-graphs, where $2 \leq p \leq 7$ and $1 \leq q \leq 11$.

### Table 2
Number of super edge-magic $(p,q)$-graphs

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7. Conclusions

With this paper, the authors hope that interest in super edge-magic labelings will be aroused among those who study graph labelings. In particular, we believe that super edge-magic labelings may provide a viable approach to the problem of showing that all trees are harmonious in the long run. In the short term, we know that they are at least useful as a means of finding graphs that are harmonious, sequential and cordial.

8. Uncited reference

[14]
Acknowledgements

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References