Anti-Magic Graphs via the Combinatorial NullStellenSatz

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Abstract: An antimagic labeling of a graph with $m$ edges and $n$ vertices is a bijection from the set of edges to the integers $1, \ldots, m$ such that all $n$ vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an antimagic labeling. In [10], Ringel conjectured that every simple connected graph, other than $K_2$, is antimagic. We prove several special cases and variants of this conjecture. Our main tool is the Combinatorial NullStellenSatz (cf. [1]). © 2005 Wiley Periodicals, Inc. J Graph Theory 50: 263–272, 2005

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1. INTRODUCTION

An antimagic labeling of a graph with $m$ edges and $n$ vertices is a bijection from the set of edges to the integers $1, 2, \ldots, m$ such that all $n$ vertex sums are pairwise distinct.
distinct, where a vertex sum is the sum of labels of all edges incident with that vertex. A graph is called antimagic if it has an antimagic labeling. A \( k \)-antimagic labeling of a graph, where \( k \) is a non-negative integer, is an injection from the set of edges to the integers \( 1, 2, \ldots, m + k \) such that all \( n \) vertex sums are pairwise distinct. A graph is called \( k \)-antimagic, abbreviated to \( k \)-AM, if it has a \( k \)-antimagic labeling (thus antimagic is the same as \( 0 \)-AM).

A \((\omega, k)\)-antimagic labeling of a graph, where \( k \) is a non-negative integer and \( \omega : V \rightarrow \mathbb{N} \) is a weight function on the set of vertices, is an injection from the set of edges to the integers \( 1, 2, \ldots, m + k \) such that all \( n \) vertex sums are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with that vertex and its initial weight under \( \omega \). A graph is called \((\omega, k)\)-antimagic, abbreviated to \((\omega, k)\)-AM, if it has a \((\omega, k)\)-antimagic labeling (thus antimagic is the same as \((0, 0)\)-AM, where the first 0 is the zero function).

An oriented antimagic labeling of a digraph with \( m \) edges and \( n \) vertices is a bijection from the set of edges to the integers \( 1, 2, \ldots, m \) such that all \( n \) oriented vertex sums are pairwise distinct, where an oriented vertex sum is the sum of labels of the edges entering that vertex minus the sum of labels of the edges leaving it. A digraph is called antimagic if it has an oriented antimagic labeling.

If in an antimagic labeling of \( G = (V, E) \), the vertex sums are pairwise distinct modulo \( |V| \), then we say that \( G \) is edge graceful (this concept was introduced in [11]). In [10], Ringel conjectured that every simple connected graph, other than \( K_s \), is antimagic. We prove several special cases and variants of this conjecture. Our main results are the following:

**Theorem 1.1.** Let \( G \) be a graph on \( n = 3^k \) vertices, \( k \in \mathbb{N} \). If \( G \) admits a \( K_3 \)-factor then it is antimagic.

**Theorem 1.2.** Let \( G = (V, E) \) be a graph on \( n \) vertices and \( m \) edges, such that \( G = H \cup f_1 \cup \ldots \cup f_r \), where \( H = (V, E) \) is edge graceful and the \( f_i \)'s are 2-factors; then \( G \) is edge graceful.

It is known (cf. [5]) that a \( 2d \)-regular graph can be decomposed into 2-factors, thus we have the following corollary:

**Corollary 1.3.** Let \( G \) be a \( 2d \)-regular graph. If \( G \) has a 2-factor consisting of \( k \) circuits, each of length \( t \), where \( k, t \) are odd integers then \( G \) is edge graceful.

Note that any 2-factor on an even number of vertices is not edge graceful as if \( n \) is even \( \sum_{i=0}^{n-1} i \neq 0 \mod n \) but \( 2 \sum_{i=0}^{n-1} i = 0 \mod n \), so the vertex sums cannot all be different modulo \( n \).

We will also prove the following results:

**Theorem 1.4.** Every graph with at most one isolated vertex and no isolated edges is \((\omega, 2|V| - 4)\)-AM for every initial weight function \( \omega \).

For certain families of graphs, we can do better than \( 2|V| - 4 \):

**Theorem 1.5.** If \( G = (V, E) \), where \( |V| > 2 \), admits a 1-factor then it is \((|V| - 2)\)-AM.
Theorem 1.6. If $G = (V, E)$ is a $(2d + 1)$-regular bipartite graph, where $d \geq 1$, then:

(a) If moreover, there exists a decomposition of $G$ into a 1-factor and $d$ 2-factors whose every circuit is of length devisable by 4 then $G$ is $(\binom{n}{2} - 1)$-AM.

(b) $G$ has an orientation such that the resulting digraph is antimagic.

In [3], the authors prove that a graph $G$ on $n$ vertices with maximal degree $\leq n - 2$ is antimagic. They ask if the same can be proved for graphs with maximal degree $n - k$ where $k \geq 3$ and $n$ depends on $k$. We prove the following:

Theorem 1.7. Let $G$ be a graph on $n$ vertices and maximal degree $n - k$, where $k \geq 3$ is any function of $n$; then

(a) $G$ is $(3k - 7)$-AM.

(b) If $n \geq 6k^2$ then $G$ is $(k - 1)$-AM.

The main tool used in this paper is the following theorem due to N. Alon (cf. [1], a similar approach appeared also in [2] and [4]):

Theorem 1.8 (Combinatorial Nullstellensatz). Let $\mathbb{F}$ be an arbitrary field, and let $f(x_1, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \ldots, x_n]$. Suppose the degree $\deg(f)$ of $f$ is $\sum_{i=1}^{n} t_i$, where each $t_i$ is a non-negative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in $f$ is non-zero. Then, if $S_1, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $f(s_1, \ldots, s_n) \neq 0$.

2. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1

In order to prove Theorem 1.1, we will need the following lemma from [12]:

Lemma 2.1. If $P(x_1, x_2, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ is of degree $\leq s_1 + s_2 + \cdots + s_n$, where $n$ is a positive integer and $s_1, s_2, \ldots, s_n$ are non-negative integers, then

$$\left( \frac{\partial}{\partial x_1} \right)^{s_1} \left( \frac{\partial}{\partial x_2} \right)^{s_2} \cdots \left( \frac{\partial}{\partial x_n} \right)^{s_n} P(x_1, x_2, \ldots, x_n)$$

$$= \sum_{x_1=0}^{s_1} \cdots \sum_{x_n=0}^{s_n} (-1)^{s_1+x_1} \binom{s_1}{x_1} \cdots (-1)^{s_n+x_n} \binom{s_n}{x_n} P(x_1, x_2, \ldots, x_n)$$

Let $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$, $n = 3k$, be a graph that admits a $K_3$-factor $f = (V, E)$, and let $r = n/3$, that is, $r$ is the number of triangles in $f$. Label the edges of $E \setminus E$ arbitrarily using labels from the set $\{n + 1, \ldots, |E|\}$. For every vertex $v \in V$, denote its current vertex sum by $\omega(v)$. It suffices to prove that $f$ is $(\omega, 0)$-AM. We will prove this to be true for any weight function $\omega$. 
Associate with \( f \) and \( \omega \) the polynomial \( Q_\omega(x_1, \ldots, x_n) = \prod_{i \geq j \geq 1}(x_i - x_j) \)
\( P_\omega(x_1, \ldots, x_n) = \prod_{i \geq j \geq 1}(x_i + x_j + \omega(v_j) - x_j - x_j - \omega(v_j)) \), where \( P_\omega(x_1, \ldots, x_n) \) represent the edges of \( f \) incident with \( v_i \) and \( x_i, x_j \) represent the edges of \( f \) incident with \( v_j \). Clearly, \( f \) is \((\omega, 0)\)-AM if and only if there exists \( a_1, \ldots, a_n \in \{1, \ldots, n\} \) such that \( Q_\omega(a_1, \ldots, a_n) \neq 0 \). By Theorem 1.8, it suffices to prove that in the expansion of \( Q_\omega(x_1, \ldots, x_n) \) there exists a monomial \( c \prod_{i=1}^n x_i^{n-1} \) such that \( c \neq 0 \). The coefficient \( c \) is equal to the coefficient of the same monomial in the expansion of \( Q_\omega(x_1, \ldots, x_n) = \prod_{i \geq j \geq 1}(x_i - x_j) \)
\( P_\omega(x_1, \ldots, x_n) \), where \( P_\omega(x_1, \ldots, x_n) = \prod_{i \geq j \geq 1}(x_i + x_j - x_i - x_j - \omega(v_j)) \). Using lemma 2.1, we get

\[
c[(n-1)!]^n = \left( \frac{\partial}{\partial x_1} \right)^{n-1} \left( \frac{\partial}{\partial x_2} \right)^{n-1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{n-1} Q_0(x_1, x_2, \ldots, x_n)
= \sum_{x_1=0}^{n-1} \cdots \sum_{x_n=0}^{n-1} (-1)^{x_1+x_2+\cdots+x_n} \binom{n-1}{x_1} \cdots \binom{n-1}{x_n} Q_0(x_1, x_2, \ldots, x_n)
= \sum_{\sigma \in S_n} (-1)^{\sigma(0)+\sigma(1)+\cdots+\sigma(n-1)} \binom{n-1}{\sigma(0)} \cdots \binom{n-1}{\sigma(n-1)}
\times Q_0(\sigma(0), \ldots, \sigma(n-1))
= (-1)^{\binom{n}{2}} \binom{n-1}{0} \cdots \binom{n-1}{n-1} \prod_{i \geq j \geq 1} (i-j)
\times \sum_{\sigma \in S_n} \text{sign}(\sigma) P_0(\sigma(0), \ldots, \sigma(n-1))
\]

It is therefore sufficient to prove that \( \sum_{\sigma \in S_n} \text{sign}(\sigma) P_0(\sigma(0), \ldots, \sigma(n-1)) \neq 0 \).

Let \( \text{Aut}(f) \) denote the automorphism group of \( f \) and let \( H = S_n/\text{Aut}(f) \). Let \( \pi \in \text{Aut}(f) \), then permuting the labels assigned to the edges of \( G, \) according to \( \pi, \) yields a permutation of the vertex sums of \( V \); we will denote this permutation by \( \pi_v \) (note that in fact, \( \text{Aut}(f) \) represents here the automorphism group of the line graph of \( f \), which is isomorphic to \( f \)). Let \( \sigma, \tau \) be in the same coset of \( H, \) i.e., \( \sigma = g\tau \) for some \( g \in \text{Aut}(f) \). \( \sigma \) and \( \tau \) yield the same vertex sums and so \( \text{sign}(\sigma) P_0(\sigma(0), \ldots, \sigma(n-1)) = \text{sign}(g) \text{sign}(\tau) \text{sign}(g_v) P_0(\tau(0), \ldots, \tau(n-1)) = \text{sign}(\tau) P_0(\tau(0), \ldots, \tau(n-1)) \), where the last equality follows since \( \text{sign}(g) = \text{sign}(g_v) \). Thus

\[
\sum_{\sigma \in S_n} \text{sign}(\sigma) P_0(\sigma(0), \ldots, \sigma(n-1)) = (3!)^r r! \sum_{[\tau] \in H} \text{sign}(\tau) P_0(\tau(0), \ldots, \tau(n-1))
\]

where \( \tau \) is any representative of its coset. Let \( S(\pi) = (P_0(\pi(0), \ldots, \pi(n-1))/ \prod_{i \geq j \geq 1}(i-j)) \mod 3 \). It is known that \( \prod_{i \geq j \geq 1}(i-j) | P_0(a_1, \ldots, a_n) \) for every \( a_1, \ldots, a_n \in \mathbb{Z} \) (cf. [8]) and so \( S(\pi) \) is well defined. We shall prove that \( \sum_{[\tau] \in H} \text{sign}(\tau) S(\tau) \neq 0 \mod 3 \).
Lemma 2.2. For every $\sigma \in S_n$, let $k_\sigma$ be the smallest positive integer such that $\sigma$ and $\sigma + k_\sigma$, where $(\sigma + k_\sigma)(i) = (\sigma(i) + k_\sigma) \mod n$ for every $0 \leq i \leq n-1$, are in the same coset of $H$. Then $k_\sigma \neq 0 \mod 3$ if and only if $\pi$ is in the same coset of $H$ as the permutation $\sigma = (0, r, 2r, 1, r + 1, 2r + 1, \ldots, r - 1, 2r - 1, 3r - 1)$.

**Proof of Lemma 2.2.** Clearly if $\pi$ is in the same coset of $H$ as $\sigma$ then $k_\pi = 1$. Let $\pi \in S_n$. The edges of some triangle are labeled $0, a, b$ where $0 < a < b < n$.

By the definition of $k_\pi$, $pk_\pi \mod n, (a + pk_\pi) \mod n$ and $(b + pk_\pi) \mod n$ are the labels of some triangle in $f$ for every $q \in \mathbb{N}$. Let $p$ be the smallest positive integer such that $\{0, a, b\} = \{pk_\pi \mod n, (a + pk_\pi) \mod n, (b + pk_\pi) \mod n\}$; clearly $p \leq r$. If $pk_\pi \equiv 0 \mod n$ then $k_\pi \equiv 0 \mod 3$ because $p < n$. Otherwise $pk_\pi \equiv a \mod n$ (the same argument can be applied to $pk_\pi \equiv b \mod n$), and thus $\{0, a, b\} = \{a, 2a, a + b\} \mod n$, where $0 < a$ and so $a + b \equiv 0 \mod n$ and $b \equiv 2a \mod n$. It follows that this triangle is labeled $0, r, 2r$. Thus if $k_\pi \neq 0 \mod 3$, i.e., $\gcd(3, k_\pi) = 1$, then this labeling is in the same coset of $H$ as $\sigma$.

Lemma 2.3. Let $\sigma$ denote the permutation $(0, r, 2r, 1, r + 1, 2r + 1, \ldots, r - 1, 2r - 1, 3r - 1)$, then $S(\sigma) \neq 0$.

**Proof of Lemma 2.3.** In order to prove the lemma, it suffices to show that for every $1 \leq t \leq k$ (recall that $n = 3^k$) and $0 \leq i, j \leq 3^t - 1$, the number of vertex sums from $\sigma_v = (r, 2r, 3r, r + 2, 2r + 2, 3r + 2, \ldots, 3r - 2, 4r - 2, 5r - 2)$ with remainder $i \mod 3^t$ is equal to the number of those with remainder $j \mod 3^t$. Since this is the case with $0, 1, \ldots, n - 1$, it will follow that $3^t \mid P_0(\sigma(0), \ldots, \sigma(n - 1))$ if and only if $3^t \mid \prod_{0 \leq i < j \leq 1}(i - j)$ for all $l \in \mathbb{N}$, and thus $S(\sigma) \neq 0$. We proceed by induction on $k$.

If $k = 1$ then $\sigma_v$ is a permutation of $1, 2, 3$ for every $\pi \in S_3$. Let $n = 3^{k+1}$. Let $R_i i = 0, 1, 2$ be the set of vertex sums of $\sigma_v$ with remainder $i \mod 3$, then $|R_0| = |R_1| = |R_2|$ (as $r, 2r, 3r = 0 \mod 3, r + 2, 2r + 2, 3r + 2 = 2 \mod 3, r + 4, 2r + 4, 3r + 4 = 1 \mod 3$ and so on). Let $A_0 = \{m/3 | m \in R_0\}, A_1 = \{(m - 4)/3 | m \in R_1\}, A_2 = \{(m - 2)/3 | m \in R_2\}$, then $A_0 = A_1 = A_2$ is the set of vertex sums obtained from $\sigma$ if $n = 3^k$. Let $t$ be an integer, $2 \leq t \leq k + 1$. By the induction hypothesis, the number of elements of $A_0$ with remainder $i \mod 3^{t-1}$ is equal to the number of elements with remainder $j \mod 3^{t-1}$ for all $0 \leq i, j \leq 3^{t-1} - 1$. It follows that this is true for $R_0 \cup R_1 \cup R_2$ modulo $3^t$. 

We claim that $S(\pi + l) = S(\pi)$ for every $\pi \in S_n$ and $l \in \mathbb{N}$. It is clearly sufficient to prove this for $l = 1$. By adding 1 to every element of $\pi$ and then subtracting $n$ from one such element (since the addition is modulo $n$), we add 2 to every element of $\pi_v$ and then subtract $n$ from two of them. Since $n = 0 \mod 3^t$ for every $1 \leq t \leq k$, this does not change the number of remainders modulo $3^t$ for every $1 \leq t \leq k$. Furthermore $\text{sign}(\pi + l) = \text{sign}(\pi)$ for every $\pi \in S_n$ and $l \in \mathbb{N}$ as $n$ is odd. By Lemma 2.2, $3 \mid k_\pi$ for every $\pi$, which is not in the same coset of $H$ as $\sigma = (0, r, 2r, 1, r + 1, 2r + 1, \ldots, r - 1, 2r - 1, 3r - 1)$, so $\sum_{[\pi] \in H} \text{sign}(\pi)S(\pi) = \text{sign}(\sigma)S(\sigma) \neq 0$, where the last inequality is by Lemma 2.3. 


Remark 2.4. If \( G = (V, E) \), where \( |V| = n \), admits a \( K_3 \)-factor, but \( 3^k < n < 3^{k+1} \), we can add \( 3^k - \frac{n}{3} \) disjoint triangles to \( G \) and then apply Theorem 1.1. Thus \( G \) is \( (3^{k+1} - n) - \text{AM} \).

Remark 2.5. Almost all the parts of the proof remain practically the same if \( G = (V, E), |V| = p^k \), admits a \( C_p \)-factor for some odd integer \( p \). The only substantial difference (which we do not know how to overcome) is that the automorphism group of \( C_p \) is not the whole of \( S_p \). For example in the automorphism group of \( C_5 \), there are 12 cosets, exactly 2 of which provide an antimagic labeling.

Proof of Theorem 1.2 and corollary 1.3. Let \( G = H \cup f_1 \cup \ldots \cup f_r \), \( |V(G)| = |V(H)| = n \), and let \( \omega \) be an edge graceful labeling of \( H \). We proceed by induction on \( r \):

If \( r = 0 \) then \( G = H \) and so \( \omega \) is an edge graceful labeling of \( G \). For \( r > 0 \), remove a 2-factor \( f_r \) from \( G \). By the induction hypothesis, \( G' = G \setminus f_r \) is edge graceful; let \( \omega' \) be an edge graceful labeling of \( G = (V, E) \). We will obtain an edge graceful labeling of \( G \) using the labels \( |E'| + 1, \ldots, |E'| + n \), which modulo \( n \) are the same as \( 0, 1, \ldots, n - 1 \). If the vertex sums, in \( G' \), of some circuit \( v_1 e_1 v_2 e_2 \cdots v_k e_k v_1 \) of \( f_r \) are \( a_1, \ldots, a_k \), then for \( 1 \leq i \leq k \), we label the edge \( e_i \) with \( b_i \) which is the inverse of \( a_i \) in \( \langle \mathbb{Z}_n, + \rangle \). Clearly the new vertex sum of \( v_1, 1 \leq i \leq k \), is \( b_{(i-1)\mod k} \).

Doing the same with every circuit of \( f_r \) yields an edge graceful labeling of \( G \).

In order to prove Corollary 1.3, it suffices to show that \( G = t \ast C_k \) is edge graceful for every odd \( k, t \). Indeed if we label the \( i \)-th circuit \( (1 \leq i \leq t) \) with \( i - 1, t + i - 1, 2t + i - 1, \ldots, (k - 1)t + i - 1 \) in that order, we get an edge graceful labeling of \( G \).

3. PROOFS OF THE REMAINING RESULTS

Proof of Theorem 1.4 (this proof is due to N. Alon). Let \( \{e_1, e_2, \ldots, e_m\} \) be an arbitrary ordering of the edges of \( G \). At stage \( i \), we will label edge \( e_i \) with an unused label from the set \( \{1, 2, \ldots, m + 2n - 4\} \) such that the current weight of both its endpoints will differ from the current weight of any other vertex. This is possible since for each endpoint, we have at most \( n - 2 \) forbidden labels and so at any stage, we have at most \( 2n - 4 \) forbidden labels and at least \( 2n - 3 \) unused labels. Since there is at most one isolated vertex and no isolated edges, this yields an antimagic labeling of \( G \).

Note that the same proof (with minor changes) applies to multi-graphs (with no connected components isomorphic to \( C_k^{\ast} \) for \( k = 1, 2, \ldots \) where \( G^{\ast} \) is the dual of \( G \)), and to digraphs.

The following lemma, which is a special case of the Dyson conjecture [6] (proved in [9], [13] and later in [7] and [14]), will be very useful in the rest of this section.
Lemma 3.1. For every positive integers \( k, n \) let \( c_{k,n} \) be the coefficient of \( \prod_{i=1}^{n} x_i^{k(n-1)} \) in \( V_n^k(x_1, \ldots, x_n) = \prod_{i \geq 1, j \geq 1} (x_i - x_j)^{2k} \); then \( c_{k,n} \neq 0 \).

For the sake of completeness, we include a short proof:

For every \( \sigma = (\sigma(1), \ldots, \sigma(n)) \), let \( n + 1 - \sigma = (n + 1 - \sigma(1), \ldots, n + 1 - \sigma(n)) \). Clearly \( \sigma \in S_n \) if and only if \( n + 1 - \sigma \in S_n \). Furthermore, \( \text{sign}(n + 1 - \sigma) = (-1)^{\left( \sum_{i=1}^{n} \text{sign}(\sigma_i) \right) / 2} \) for all such \( \sigma \). Denote by \( \text{opcr}(k, n) \) (respectively \( \text{ecr}(k, n) \)), the set of all signed involutions \( \left( \sigma_1, \sigma_2, \ldots, \sigma_{2n} \right) \in S_n^{2k} \) such that \( \sum_{i=1}^{2k} \sigma_i(j) = k(n + 1) \) for every \( 1 \leq j \leq n \) and its sign \( \prod_{i=1}^{2k} \text{sign}(\sigma_i) \) is even (respectively odd), then \( c_{k,n} = |\text{ecr}(k, n)| - |\text{opcr}(k, n)| \).

Define a function \( f : S_n^{2k} \rightarrow S_n^k \) by \( f(\sigma_1, \ldots, \sigma_k) = (n + 1 - \sigma_1, \ldots, n + 1 - \sigma_k) \). \( f \) is an involution and is therefore bijective. Moreover, it multiplies the sign of \( \sigma_1, \ldots, \sigma_k \) by \( (-1)^{\left( \sum_{i=1}^{k} \text{sign}(\omega_i) / 2 \right)} \) and so up to its sign \( c_{k,n} \) is a sum of squares. Clearly the coefficients of \( \prod_{i=1}^{n} x_i^k \) and \( \prod_{i=1}^{n} x_i^{k(n-1)} \) do not vanish and so the lemma follows.

Proof of Theorem 1.5. Let \( G \) be a graph on \( 2n \) vertices, and \( M = \{(u_i, v_i) : 1 \leq i \leq n\} \) a 1-factor of \( G \). The \( m-n \) edges of \( G \backslash M \) can be labeled such that the vertex sum of \( u_i \) differs from the vertex sum of \( v_i \) for all \( 1 \leq i \leq n \), using the integers \( 1, \ldots, m - n + 2 \). This is done as in the proof of Theorem 1.4. For every vertex \( v \) of \( G \), denote its weight under this labeling by \( \omega(v) \). Now we label the edges of \( M \). Let \( x_i \) be the label of the edge \( (u_i, v_i) \). Since we want all vertex sums to be distinct, we need \( x_i + \omega(u_i) \neq x_j + \omega(u_j) \), \( x_i + \omega(v_i) \neq x_j + \omega(v_j) \), \( x_i + \omega(u_i) \neq x_j + \omega(v_j) \), \( x_i + \omega(v_i) \neq x_j + \omega(u_j) \) for every \( 1 \leq i < j \leq n \). Since we also want \( \omega(u_i) \neq \omega(v_i) \), which is true by our labeling of \( G \backslash M \). To achieve this, one should add the constraint \( x_i \neq x_j \) for every \( 1 \leq i < j \leq n \) in order to obtain an injective labeling. It is therefore sufficient to prove that there exists a vertex \( \bar{x} = (x_1, \ldots, x_n) \) such that for every \( 1 \leq i \leq n \), \( x_i \) is taken from the set of unused labels of \( \{1, \ldots, m+2n-2\} \), and \( P_M(\bar{x}) = \prod_{i>j} (x_i - x_j)(x_i - x_j + \omega(u_i) - \omega(u_j))[x_i - x_j + \omega(v_i) - \omega(v_j)](x_i - x_j + \omega(u_i) - \omega(u_j)) \neq 0 \). By Theorem 1.8, we need to show that there exists a non-vanishing monomial \( \prod_{i=1}^{n} x_i^{k_i} \) in \( P_M \), such that \( \sum_{i=1}^{n} t_i = \deg(P_M) \) and \( t_i < 3n - 2 \) for every \( 1 \leq i \leq n \). Clearly this is the same as finding such a monomial in \( V_n^5(\bar{x}) = \prod_{i>j} (x_i - x_j)^5 \), but by Lemma 3.1, this is true even for \( V_n^6 \).

Proof of Theorem 1.6. (a) Let \( G = (U \cup V, E) \) be a bipartite graph with \( U = \{u_1, \ldots, u_n\} \) and \( V = \{v_1, \ldots, v_n\} \). Let \( \{(u_i, v_i) : 1 \leq i \leq n\} \) be a 1-factor and let \( \{f_i : 1 \leq i \leq d\} \) be 2-factors with no circuits of length 2 mod 4. Label \( f_i \) with \( 1, 2dn, 2, 2dn - 1, \ldots, n, 2dn + 1 \) and the other 2-factors \( \{f_i : 2 \leq i \leq d\} \) with \( n(i-1)+1, 2dn-n(i-1), 2dn-n(i-1)-1, n(i-1)+2, n(i-1)+3, \ldots \).
\[ \ldots, n(i-1) + n, 2dn - n(i-1) - n + 1, \] respectively, in both cases starting with some arbitrary edge \((u_k, v_j)\). The sum at all vertices of \(V\) is \(d(2dn + 1)\) (\(2dn + 1\) from any 2-factor), and the sum at all vertices \(U\) is not \(d(2dn + 1)\) (in fact, it has the opposite parity). Denote the current weight of a vertex \(v\) by \(\omega(v)\) and the weights of the edges \(\{(u_i, v_j)| 1 \leq i \leq n\}\) by \(x_1, \ldots, x_n\), respectively. As in the proof of Theorem 1.5, we want \(x_i + \omega(u_k) \neq x_j + \omega(u_k), x_i + \omega(u_k) \neq x_j + \omega(u_k)\) for every \(1 \leq i < j \leq n\). Since \(\omega(v_1) = \cdots = \omega(v_n)\), the extra constraint \(i \neq j \Rightarrow x_i \neq x_j\) is redundant. Hence we need a non-vanishing monomial of degree \(<2n-1\) in \(V_n^2\) whose existence is guaranteed by lemma 3.1.

(b) Direct all the edges from \(U\) to \(V\), and remove a perfect matching \(\{(u_i, v_j)| 1 \leq i \leq n\}\). Using the same labeling as in (a) we get a vertex sum of \(d(2dn + 1)\) at every vertex of \(V\), and a non-positive vertex sum in every vertex of \(U\). Regardless of the weights of \((u_i, v_j)\), the oriented vertex sum of every \(v_i\) will be positive and that of every \(u_i\) negative, so we require only \(x_i \neq x_j, x_i + \omega(u_k) \neq x_j + \omega(u_k)\) for every \(1 \leq i < j \leq n\). Thus we need a non-vanishing monomial of degree \(<n\) in \(V_n^2\) whose existence is guaranteed by lemma 3.1. ■

**Proof of Theorem 1.7.** (a) Let \(G = (V, E)\), where \(|V| = n, |E| = m\), be a graph of maximal degree \(n-k\), \(k \geq 3\). Let \(v \in V\) be a vertex of degree \(n-k\), \(v_1, \ldots, v_{n-k}\) its neighbors, and \(A = \{u_1, \ldots, u_{k-1}\}\) the other vertices of \(G\). Let \(E = E_1 \cup E_2 \cup E_3\), where \(E_1 = \{(v, v_i)| 1 \leq i \leq n-k\}\), \(E_2\) is the set of edges with at least one endpoint in \(A\), and \(E_3\) is the (possibly empty) set of the remaining edges. First we give a labeling to the edges of \(E_2\), using the smallest possible labels, such that the vertex sums of \(u_1, \ldots, u_{k-1}\) are pairwise distinct. This can be done by applying the same method as in the proof of Theorem 1.4. If we label an edge \((u_i, u_j)\) then we have \(2k-6\) forbidden labels. If we label an edge \((u_i, v_j)\) then we have only \(k-2\) forbidden labels (for \(u_i\)). Thus we can achieve this, using \(|E_2|\) labels from the set \(\{1, 2, \ldots, |E_2| + 2k - 6\}\) (if \(k \geq 4\) then \(2k - 6 \geq k - 2\) and if \(k = 3\) then we can verify that \(|E_2|\) labels are sufficient by a direct inspection of the various cases). Denote the resulting vertex sums of \(u_1, \ldots, u_{k-1}\) by \(a_1, \ldots, a_{k-1}\), respectively, and note that these are final regardless of the labels that will be given to the rest of the edges. Next, label the edges of \(E_3\) arbitrarily, using the smallest unused labels. Denote the current vertex sum of the \(v_i\)'s by \(\omega(v_i)\), and assume without loss of generality that \(\omega(v_1) \leq \cdots \leq \omega(v_{n-k})\). Next label the edges of \(E_1\), using the largest \(n-1\) labels \(b_1 < \cdots < b_{n-1}\) (this is possible since we have \(m + 3k - 7\) labels) as follows: label the edge \((v, v_1)\) with \(b_{i_1}\) where \(1 \leq i_1 \leq n-1\) is the smallest integer such that \(\omega(v_1) + b_{i_1} \neq a_t\) for every \(1 \leq t \leq k-1\) (such an \(i_1\) exists since we have \(n-1\) labels and only \(k-1\) restrictions). Next, do the same with \((v, v_2)\) and \(b_{i_2}\), where \(i_1 < i_2 \leq n-1\) is the smallest integer such that \(\omega(v_2) + b_{i_2} \neq a_t\) for every \(1 \leq t \leq k-1\) (this time we have \(n-k-1\) unlabelled edges, \(n-1-i_1\) labels, and at most \(k-1-(i_1-1)\) restrictions as if \(\omega(v_1) + b_{i_1} > a_t\) for some \(1 \leq t \leq k-1\) then \(\omega(v_2) + b_{i_2} > a_t\)). Hence, doing the same for the rest of the edges of \(E_1\) (after labeling the edge
(v, vj), we have n − 1 − j unlabeled edges, n − 1 − ij labels, and at most k − 1 + (j − ij) restrictions), yields a labeling ω′ such that ω′(v1) < ⋯ < ω′(vn−k) and ω′(vi) ̸= ω′(uj) for every 1 ≤ i ≤ n − k, 1 ≤ j ≤ k − 1. Finally v is of maximal degree and has the largest labels and so G is antimagic.

(b) If |E3| ≥ 2k − 6 then following the proof of (a) we see that m + k − 1 labels are enough to ensure that we will have the largest n − 1 labels for the edges of |E1| and thus v will have the largest vertex sum. If |E3| < 2k − 6 then m ≤ (n − k) + (2k − 6) + (k − 1)(n − k) ≤ kn. The edges of E1 receive at most 2k − 6 small labels (which we’ll ignore) and so at least n − 3k of these edges receive the largest labels. The degree of any other vertex is at most n − k. Let u ∈ V be a vertex of degree d. If d < n − 3k then every label assigned to u is strictly smaller then every label assigned to v and so ω(v) > ω(u). We therefore assume that n − 3k ≤ d ≤ n − k. The largest n − 3k labels assigned to u are strictly smaller than those assigned to v thus 2(ω(v) − ω(u)) ≥ (n − 3k)^2 − 4k(m + k − 1) ≥ n^2 + 9k^2 − 6kn − 4k^2n − 4k^2 + 4k > 0 as n ≥ 6k^2 and k ≥ 3.

4. CONCLUDING REMARKS AND OPEN PROBLEMS

• Note that while the proofs using the Combinatorial Nullstellensatz are non-constructive, the proofs of Theorems 1.4 and 1.7 and Corollary 1.3 provide an efficient algorithm for finding the required labeling.
• Theorems 1.4, 1.5, and 1.7 can be adapted, by introducing minor changes, to handle injections from the set of edges to any set S ⊆ R of size m + k. This fact enables us to consider operations other than vertex sums. For example, we can consider the product of all edges incident with a vertex by taking logarithms.

On the other hand, while the graphs P_i, i = 3, 4, 5 are clearly antimagic, they cannot be labelled with the numbers −1, 0, . . . , i − 3 such that the vertex sums are pairwise distinct. The same is true for K_1,n with labels −(n − 1)/2, . . . , 0, . . . , (n − 1)/2 for n ∈ N_{odd} and labels −n/2, . . . , 0, . . . , (n/2) − 1 for n ∈ N_{even}. It may be interesting to characterize all such graphs.

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REFERENCES


