Note

Bandwidth of the cartesian product of two connected graphs

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Abstract

The bandwidth $B(G)$ of a graph $G$ is the minimum of the quantity $\max\{|f(x) - f(y)| : xy \in E(G)\}$ taken over all injective integer numberings $f$ of $G$. The cartesian product of two graphs $G$ and $H$, written as $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and with $(u_1, v_1)$ adjacent to $(u_2, v_2)$ if either $u_1$ is adjacent to $u_2$ in $G$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $H$. In this paper we investigate the bandwidth of the cartesian product of two connected graphs. For a graph $G$, we denote the diameter of $G$ and the connectivity of $G$ by $D(G)$ and $\kappa(G)$, respectively. Let $G$ and $H$ be two connected graphs. Among other results, we show that if $B(H) = \kappa(H)$ and $|V(H)| \geq 2B(H)D(G) - \min\{1, D(G) - 1\}$, then $B(G \times H) = B(H)|V(G)|$. Moreover, the order condition in this result is sharp. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider finite undirected graphs without loops or multiple edges. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $D(G)$ denote the diameter of $G$, and let $\kappa(G)$ denote the connectivity of $G$. We denote the path, the cycle, the complete, and the star graph on $n$ vertices by $P_n$, $C_n$, $K_n$ and $K_{1,n-1}$, respectively. We denote the $k$th power of a graph $G$ by $G^k$. A one-to-one mapping $f : V(G) \rightarrow \mathbb{Z}$ is said to be a numbering of $G$, where $\mathbb{Z}$ stands for the set of all integers. If $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$, then $f$ is called a proper numbering of $G$. The bandwidth of a numbering $f$ of $G$,
denoted by $B_f(G)$, is the maximum difference between $f(x)$ and $f(y)$ when $xy$ runs over all the edges of $G$, namely,

$$B_f(G) = \max\{|f(x) - f(y)| : xy \in E(G)\}.$$

We define the bandwidth of $G$, the minimum of $B_f(G)$, over all numberings $f$ of $G$, and denote it as $B(G)$, i.e.,

$$B(G) = \min\{B_f(G) : f \text{ is a numbering of } G\}.$$

A numbering $f$ of $G$ is called a bandwidth numbering when $B_f(G) = B(G)$. We remark that the bandwidth of a graph $G$ is achieved by a proper numbering of $G$.

The bandwidth problem for the graphs arises from sparse matrix computation, coding theory, and circuit layout of VLSI designs. Papadimitriou [11] proved that the problem of determining the bandwidth of a graph is NP-complete, and Garey et al. [8] showed that it remains NP-complete even if graphs are restricted to trees with maximum degree 3. So many studies have been done towards finding the bandwidth of specific classes of graphs (see [2,3,10]). In this paper we investigate the bandwidth of the cartesian product of two connected graphs.

The cartesian product (or simply product) of two graphs $G$ and $H$, written as $G \times H$, is the graph whose vertex set is $V(G) \times V(H)$ with two vertices $(u_1,v_1)$ and $(u_2,v_2)$ adjacent if and only if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

There are some results on bandwidth of the product of certain graphs. For instance, $B(P_m \times P_n) = \min\{m,n\}$ if $\max\{m,n\} \geq 2$ [5] and $B(P_m \times C_n) = \min\{2m,n\}$ if $m \geq 2$ and $n \geq 3$ [6]. Moreover, Hendrich and Stiebitz [9] gave similar formulas for $B(C_m \times C_n)$, $B(K_m \times P_n)$, $B(K_m \times C_n)$, and $B(K_m \times K_n)$ (see [10]). However, these are the results for the bandwidth of the product of two specified graphs. We study the bandwidth of the product of two connected graphs which satisfy some order condition. The following upper bound on the bandwidth of the product of two graphs is known.

**Proposition 1** (Chvátalová [6]). For every two graphs $G$ and $H$,

$$B(G \times H) \leq \min\{B(H)|V(G)|, B(G)|V(H)|\}.$$ 

We get the following theorem, which gives a lower bound of the product of two connected graphs. Let $\omega(G) = \min\{1,D(G) - 1\}$, i.e., $\omega(G) = 0$ if $G$ is complete with $|V(G)| \geq 2$ and $\omega(G) = 1$ if $G$ is not complete. To state our results we use $\omega(G)$.

**Theorem 1.** Let $G$ and $H$ be two connected graphs and let $k$ be a positive integer satisfying $k \leq \kappa(H)$. If $|V(H)| \geq 2kD(G) - \omega(G)$, then

$$B(G \times H) \geq k|V(G)|.$$ 

Note that for every graph $G$, $B(G) \geq \kappa(G)$ [6]. From Theorem 1 together with Proposition 1, we get the following theorem.
Theorem 2. Let $G$ and $H$ be two connected graphs. If $B(H) = \kappa(H)$ and $|V(H)| \geq 2B(H)D(G) - \omega(G)$, then

$$B(G \times H) = B(H)|V(G)|.$$

Remark. Let $H$ be a graph on $n$ vertices. Let $k, d$ and $m$ be positive integers.

(a) If $n \leq 2k - 1$, then $B(K_{2m} \times H) \leq 2mk - 1$.

(b) If $d \geq 2, k \geq [(B(H) + 1)/2]$ and $n \leq 2kd - 2$, then there are infinitely many graphs $G$ such that $D(G) = d$ and $B(G \times H) \leq k|V(G)| - 1$.

This remark assures us that if $D(G) = 1$ or $[(B(H) + 1)/2] \leq k \leq \kappa(H)$, then the order condition in Theorem 1 is sharp, and hence the order condition in Theorem 2 is also sharp.

Let $H$ be a graph on $n$ vertices. Then $B(H) = \min\{k : H \subseteq P_n^k\}$ (see [2,3]). Therefore, $B(H) = \kappa(H) = k$ if and only if $H \subseteq P_n^k$ and $\kappa(H) = k$. For instance, $B(P_1^1) = \kappa(P_n^1) = l$ ($n \geq l + 1$), $B(C_4^1) = \kappa(C_4^1) = 2l$ ($n \geq 2l + 1$), and $B(K_m \times P_n) = \kappa(K_m \times P_n) = m$ ($n \geq 2$). Hence, we obtain the following corollary.

Corollary 1. Let $G$ be a connected graph. Let $l, m$ and $n$ be positive integers.

(i) If $n \geq 2|D(G) - \omega(G)|$, then $B(G \times P_n^l) = l|V(G)|$.

(ii) If $n \geq 4|D(G) - \omega(G)|$, then $B(G \times C_n^l) = 2l|V(G)|$.

(iii) If $mn \geq 2m|D(G) - \omega(G)|$, then $B(G \times K_m \times P_n) = m|V(G)|$.

Moreover, the above order conditions are sharp.

We give the bandwidth of the product of $K_{1,m}$ and $P_n$. We verify that $B(K_{1,m}) = \lceil m/2 \rceil$ and $B(K_{1,1} \times P_2) = B(K_{1,2} \times P_2) = 2$. A theta graph consists of a pair of vertices and $m$ internally vertex-disjoint paths between them. For $m = 1$ and $2$, a theta graph is a path and a cycle, respectively. Let $\Theta_m$ be the theta graph where all $m$ paths have length $3$. Chvátalová and Opatrný [7] proved that $B(\Theta_m) = B(K_{1,m} \times P_3) = [(3m + 2)/4]$ for $m \geq 3$. By Theorem 2, we have $B(K_{1,m} \times P_3) = m + 1$ if $n \geq 3$. Therefore, we get the following corollary.

Corollary 2.

$$B(K_{1,m} \times P_n) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil & \text{if } n = 1, \\ \left\lceil \frac{3m + 2}{4} \right\rceil & \text{if } n = 2, \\ m + 1 & \text{if } n \geq 3. \end{cases}$$

This paper consists of three sections. In Section 2, we show the above remark and preliminary results to prove Theorem 1. In Section 3, we prove Theorem 1.
2. Preliminaries

Let \( G \) be a graph. For a vertex \( x \in V(G) \), let \( N_G(x) \) denote the neighborhood of \( x \) in \( G \), i.e., \( N_G(x) \) is the set of vertices adjacent to \( x \) in \( G \), and let \( N_G[x] \) denote \( N_G(x) \cup \{x\} \). More generally, for every \( X \subseteq V(G) \), we define \( N_G(X) = \bigcup_{x \in X} N_G(x) \) and \( N_G[X] = N_G(X) \cup X \). For two vertices \( x, y \in V(G) \), \( d_G(x, y) \) denotes the distance between \( x \) and \( y \) in \( G \), and for a vertex \( x \in V(G) \) and a subset \( Y \subseteq V(G) \), we define \( d_G(x, Y) = \min\{d_G(x, y) : y \in Y\} \).

For a subset \( X \) of \( V(G) \), let \( G[X] \) denote the graph induced by \( X \) in \( G \), and let \( G - X \) denote the graph obtained from \( G \) by deleting \( X \); thus, \( G - X = G[V(G) - X] \). We denote the edge independence number of \( G \) by \( \alpha_1(G) \). The following propositions are used in the proofs that follow.

**Proposition 2** (Chvátal [4]). Let \( G \) be a graph and let \( f \) be a bandwidth numbering of \( G \). Then
\[
|f(u) - f(v)| \leq B(G)d_G(u, v) \quad \text{for } u, v \in V(G).
\]

**Proposition 3** (Berge [1, p. 132]). Let \( G \) be a bipartite graph with partite sets \( X \) and \( Y \). Then
\[
\alpha_1(G) = |X| - \max_{A \subseteq X} \{|A| - |N_G(A)|\}.
\]

Let \( G \) and \( H \) be two graphs on \( m \) and \( n \) vertices, respectively. Write \( V(H) = \{v_1, v_2, \ldots, v_n\} \). We define \( G_i = (G \times H)[V(G) \times \{v_i\}] \) for \( i = 1, 2, \ldots, n \). For \( X \subseteq V(G \times H) \), we define \( X_i = X \cap V(G_i) \) for \( i = 1, 2, \ldots, n \), and \( X_0 = \{v_i : X_i = \emptyset\} \). First we show Remark(a) in Section 1.

**Lemma 1.** Let \( H \) be a graph on \( n \) vertices. Then
\[
B(K_{2m} \times H) \leq mn + m - 1.
\]

**Proof.** Let \( g \) be a proper bandwidth numbering of \( H \). Let \( V(K_{2m}) = \{u_1, u_2, \ldots, u_{2m}\} \), \( V(H) = \{v_1, v_2, \ldots, v_n\} \), and let \( G = K_{2m} \). Consider a numbering \( f \) of \( G \times H \) defined as follows: For \( i = 1, 2, \ldots, n \),
\[
f(V(G_i)) = \{(g(v_i) - 1)m + j, (g(v_i) - 1)m + mn + j) : j = 1, 2, \ldots, m\}
\]
and \( f(u_{j_1}, v_i) < f(u_{j_2}, v_i) \) if \( 1 \leq j_1 < j_2 \leq 2m \).

Then, for \( w_1w_2 \in E(G \times H) \),
\[
|f(w_1) - f(w_2)| \leq \begin{cases} 
mn + m - 1 & \text{if } w_1w_2 \in E(G_i), \\
mB(H) & \text{otherwise.} 
\end{cases}
\]

Therefore, by \( B(H) \leq n - 1 \), we get \( B(K_{2m} \times H) \leq mn + m - 1 \). \( \square \)
From Lemma 1, we can obtain Remark(a) in Section 1. Next we show Remark(b) in Section 1. Let \(d \geq 2\) and \(r \geq 1\) be integers. We define a graph \(G(d, r)\) as follows:

\[
\begin{align*}
V(G(d, r)) &= \{x_1, x_2, \ldots, x_r\} \cup \{y_1, y_2, \ldots, y_r\} \cup \{z_1, z_2, \ldots, z_{d-1}\}, \\
E(G(d, r)) &= \{x_i z_i, y_i z_{d-1} : i = 1, 2, \ldots, r\} \cup \{z_i z_{i+1} : i = 1, 2, \ldots, d-2\}.
\end{align*}
\]

We remark that \(G(2, r)\) is a star and \(G(3, r)\) is a double star. Furthermore, note that \(|V(G(d, r))| = 2r + d - 1\) and \(D(G(d, r)) = d\).

**Lemma 2.** Let \(H\) be a graph on \(n\) vertices. Then

\[
B(G(d, r) \times H) \leq \max \left\{ (r + d - 1) \left[ \frac{n + 2}{d} \right] - \left\lfloor \frac{r}{d} \right\rfloor - 1, \quad (r + d - 1)B(H) \right\}
\]

**Proof.** Let \(g\) be a proper bandwidth numbering of \(H\). Write \(V(H) = \{v_1, v_2, \ldots, v_n\}\). Let \(X = \{x_1, x_2, \ldots, x_r\} \times V(H)\), \(Y = \{y_1, y_2, \ldots, y_r\} \times V(H)\), and \(Z = \{z_1, z_2, \ldots, z_{d-1}\} \times V(H)\). Let \(p = r + d - 1\). Consider a numbering \(f\) of \(G(d, r) \times H\) defined as follows: For \(i = 1, 2, \ldots, n\),

\[
\begin{align*}
f(X_i) &= \{(g(v_i) - 1)p + j : j = 1, 2, \ldots, p\} \\
&\quad - \left\{ (g(v_i) - 1)p + \left\lfloor \frac{j}{d} \right\rfloor + j : j = 1, 2, \ldots, d - 1 \right\} \\
(f(x_{j_1}, v_i) < f(x_{j_2}, v_i) \text{ if } 1 \leq j_1 < j_2 \leq r),
\end{align*}
\]

\[
f(Y_i) = \{f(x_j, v_i) + np : j = 1, 2, \ldots, r\} \quad (f(y_{j_1}, v_i) < f(y_{j_2}, v_i) \text{ if } 1 \leq j_1 < j_2 \leq r)
\]

and

\[
f(Z_i) = \left\{ (g(v_i) + \left\lfloor \frac{(n + 2)f}{d} \right\rfloor - 2)p + \left\lfloor \frac{(d - f)r}{d} \right\rfloor + d - j : j = 1, 2, \ldots, d - 1 \right\}
\]

\[
(f(z_{j_1}, v_i) < f(z_{j_2}, v_i) \text{ if } 1 \leq j_1 < j_2 \leq d - 1).
\]

Fig. 1 illustrates a numbering \(f\) of \(G(3, 6) \times P_4\). Then for \(w_1w_2 \in E(G(d, r) \times H)\),

\[
|f(w_1) - f(w_2)| \leq \begin{cases} (r + d - 1) \left[ \frac{n + 2}{d} \right] - \left\lfloor \frac{r}{d} \right\rfloor - 1 & \text{if } w_1w_2 \in E(G(d, r)), \\
(r + d - 1)B(H) & \text{otherwise.}
\end{cases}
\]

Therefore, we obtain this lemma. \(\Box\)

From Lemma 2, we can get the following corollary, which shows Remark(b) in Section 1.
Corollary 3. Let $H$ be a graph on $n$ vertices. If $d \geq 2$, $k \geq \lceil (B(H) + 1)/2 \rceil$, $r \geq kd(d-1)$ and $n \leq 2kd - 2$, then $B(G(d,r) \times H) \leq k|V(G(d,r))| - 1$.

To prove Theorem 1, we need the following lemma.

Lemma 3. Let $G$ and $H$ be two connected graphs on $m$ and $n$ vertices, respectively. Write $V(H) = \{v_1, v_2, \ldots, v_n\}$. Let $X, Y$ be two subsets of $V(G \times H)$ and let $k$ be a non-negative integer satisfying $k \leq \kappa(H)$. If (i) $|X_i| + |Y_j| \leq m$ when $v_i \in N_H[v_j]$ and (ii) $|X_0|, |Y_0| \geq k$, then $|X| + |Y| \leq (n - k)m$.

Proof. Let $x$ and $y$ be two distinct vertices with $x, y \notin V(G) \cup V(H)$. We define a graph $H'$ with vertex set $V(H) \times \{x, y\}$ and with edge set $\{(v_i, x)(v_j, y) : v_i \in N_H[v_j]\}$. We remark that $H'$ is a bipartite graph with partite sets $V(H) \times \{x\}$ and $V(H) \times \{y\}$. Let $S$ and $T$ be $k$-subsets of $X_0$ and $Y_0$, respectively. Let $H'' = H' - ((S \times \{x\}) \cup (T \times \{y\}))$. Since $H$ is a $k$-connected graph, this implies $|N_H[Z]| \geq \min\{n, |Z| + k\}$ for every non-empty subset $Z$ of $V(H)$. Therefore, for every $A \subseteq V(H'') \cap (V(H) \times \{x\})$, we get $|N_{H''}(A)| \geq |A|$. By Proposition 3, $H''$ has a complete matching $M$. Hence, by condition (i),

$$|X| + |Y| = \sum_{(v_i, x)(v_j, y) \in M} (|X_i| + |Y_j|) \leq (n - k)m.$$
3. Proof of Theorem 1

Let \( m = |V(G)| \), \( n = |V(H)| \), \( d = D(G) \), \( \varepsilon = \varepsilon(G) \), and \( T = G \times H \). Write \( V(G) = \{u_1, u_2, \ldots, u_m\} \) and \( V(H) = \{v_1, v_2, \ldots, v_n\} \). Let \( f \) be a proper bandwidth numbering of \( T \). We define \( S^{(i)} = f^{-1}\{1, 2, \ldots, i\} \) and \( T^{(j)} = f^{-1}\{j, j+1, \ldots, mn\} \) for \( 1 \leq i, j \leq mn \). If \( m = 1 \), then the conclusion of Theorem 1 is true. So we may assume that \( m \geq 2 \) and \( d \geq 1 \). Let \( p \) be a non-negative integer satisfying \( n = 2kd + p - \varepsilon \). By way of contradiction, assume that \( B(T) \leq km - 1 \).

**Claim 1.** If \( |S^{(i)}| + |T^{(j)}| \leq (k + p - \varepsilon)m + 2d - 1 \), then \( d_T(x, y) \geq 2d \) for all \( x \in S^{(i)} \) and all \( y \in T^{(j)} \).

**Proof.** Note that \( j > i \), since \( n = 2kd + p - \varepsilon \). By the assumption, \( |S^{(i)}| + |T^{(j)}| = i + (mn - j + 1) \leq (k + p - \varepsilon)m + 2d - 1 \), and hence \( j - i \geq mn - (k + p - \varepsilon)m - 2d + 2 = (km - 1)(2d - 1) + 1 \). Therefore, for all \( x \in S^{(i)} \) and all \( y \in T^{(j)} \), \( |f(x) - f(y)| \geq j - i \geq (km - 1)(2d - 1) + 1 \). By Proposition 2, we obtain \( (km - 1)(2d - 1) + 1 \leq B(T)d_T(x, y) \). Thus, we get \( d_T(x, y) \geq 2d \), since \( B(T) \leq km - 1 \). \( \square \)

**Claim 2.** Suppose that \( p \leq k - \varepsilon \). For every two \((k - \varepsilon)\)-subsets \( A, B \) of \( V(H) \), there exists a \((k - p - \varepsilon)\)-subset \( A' \) of \( A \) and a \((k - p - \varepsilon)\)-subset \( B' \) of \( B \) such that there exists a one-to-one mapping \( g: A' \rightarrow B' \) satisfying \( d_T(v, g(v)) \leq 2d - 1 \) for all \( v \in A' \).

**Proof.** Write \( A = \{a_1, a_2, \ldots, a_{k-\varepsilon}\} \) and \( B = \{b_1, b_2, \ldots, b_{k-\varepsilon}\} \). Let \( X = \{x_1, x_2, \ldots, x_{k-\varepsilon}\} \) and \( Y = \{y_1, y_2, \ldots, y_{k-\varepsilon}\} \) be disjoint vertex sets with \( V(H) \cap (X \cup Y) = \emptyset \). We define a graph \( H' \) with vertex set \( X \cup Y \) and with edge set \( \{x_i y_j: d_T(a_i, b_j) \leq 2d - 1\} \).
We remark that \( H' \) is a bipartite graph with partite sets \( X \) and \( Y \). Since \( H \) is a \( k \)-connected graph, this implies that \( \{|v \in V(H): d_T(v, Z) \leq r\}| \geq \min\{n, |Z| + rk\} \) for every non-empty subset \( Z \) of \( V(H) \). Therefore, for all non-empty subsets \( S \) of \( X \), we get \( |N_{H'}(S)| \geq \min\{n, |S| + (2d - 1)k\} - |V(H) - B| \), and it follows that \( |S| - |N_{H'}(S)| \leq p \). Hence, from Proposition 3, we have \( x_i(H') \geq k - p - \varepsilon \). Thus, the conclusion of this claim is true. \( \square \)

We define \( \alpha = \min\{i: |V(H) - S^{(i)}_e| = k\} \) and \( \beta = \max\{j: |V(H) - T^{(j)}_e| = k\} \). We remark that \( |S^{(\alpha)}| + |T^{(\beta)}| \leq (k - 1)m + 1 \).

**Claim 3.** If \( d = 1 \) and \( p \leq k \), then \( |S^{(\alpha)}| + |T^{(\beta)}| \leq (k + p)m \).

**Proof.** Note that \( \varepsilon = 0 \), since \( d = 1 \). Let \( A = \{v_i: S^{(i)}_e \neq \emptyset\} \) and \( B = \{v_i: T^{(i)}_e \neq \emptyset\} \). We remark that \( |A| = |B| = k \). By Claim 2, there exists a \((k - p)\)-subset \( A' \) of \( A \) and a \((k - p)\)-subset \( B' \) of \( B \) such that there exists a one-to-one mapping \( g: A' \rightarrow B' \) satisfying \( d_T(v, g(v)) \leq 1 \) for all \( v \in A' \). Let \( l = \min\{i: |S^{(i)}| + |T^{(j)}_e| = (k + p)m + 1\} \). If \( l \leq \alpha \), then by Claim 1,

\[
|S^{(i)}| + |T^{(j)}| = \sum_{v_i \in A'} (|S^{(i)}| + |T^{(j)}_e|) + \sum_{v_i \in (A - A')} |S^{(i)}| + \sum_{v_i \in (B - B')} |T^{(j)}_e| \\
\leq (k - p)m + 2pm = (k + p)m,
\]
which contradicts the definition of \( l \). Therefore, we get \( l \geqslant x + 1 \), and hence \( |S^{(x)}| + |T^{(\beta)}| \leqslant (k + p)m \). \( \square \)

Claim 4. If \( d \geqslant 2 \) and \( p \leqslant k - 1 \), then \( |S^{(x)}| + |T^{(\beta)}| \leqslant (k + p - 1)m + 2 \).

Proof. We argue as in Claim 3. Note that \( e = 1 \), since \( d \geqslant 2 \). If \( k = 1 \), then \( x = 1 \), \( \beta = mn \), and hence the conclusion of this claim is true. So we may assume that \( k \geqslant 2 \), \( x \geqslant 2 \), and \( \beta \leqslant mn - 1 \). Let \( A = \{ v_j; S^{(x-1)}_j \neq \emptyset \} \) and \( B = \{ v_j; T^{(\beta+1)}_j \neq \emptyset \} \). We remark that \(|A| = |B| = k - 1 \). By Claim 2, there exists a \((k - p - 1)\)-subset \( A' \) of \( A \) and a \((k - p - 1)\)-subset \( B' \) of \( B \) such that there exists a one-to-one mapping \( g : A' \rightarrow B' \) satisfying \( d_H(v, g(v)) \leqslant 2d - 1 \) for all \( v \in A' \). Let \( l = \min \{ i: |S^{(i)}| + |T^{(\beta+1)}| = (k + p - 1)m + 1 \} \). If \( l \leqslant x - 1 \), then by Claim 1,

\[
|S^{(l)}| + |T^{(\beta+1)}| = \sum_{v_i \in A'} (|S^{(l)}_i| + |T^{(\beta+1)}_i|) + \sum_{v_j \in (A-A')} |S^{(l)}_j| + \sum_{v_j \in (B-B')} |T^{(\beta+1)}_j|
\]

\[
\leqslant (k - p - 1)m + 2pm = (k + p - 1)m,
\]

which contradicts the definition of \( l \). Therefore, we obtain \( l \geqslant x \), and hence \( |S^{(x)}| + |T^{(\beta)}| = |S^{(x)}| + |T^{(\beta+1)}| + 1 \leqslant (k + p - 1)m + 2 \). \( \square \)

Claim 5. \( \beta - x > (2d - 1)(km - 1) + 1 \).

Proof. We divide our proof into three cases depending upon the value of \( d \) and \( p \).

Case 1: \( d = 1 \) and \( p \leqslant k \). By Claim 3, \(|S^{(x)}| + |T^{(\beta)}| = x + (mn - \beta + 1) \leqslant (k + p)m \), and hence \( \beta - x \geqslant mn - (k + p)m + 1 = km + 1 > km \).

Case 2: \( d \geqslant 2 \) and \( p \leqslant k - 1 \). By Claim 4, \(|S^{(x)}| + |T^{(\beta)}| = x + (mn - \beta + 1) \leqslant (k + p - 1)m + 2 \), and it follows that \( \beta - x \geqslant mn - (k + p - 1)m + 1 = (2d - 1)(km - 1) + 1 \).

Case 3: Otherwise, \( p \geqslant k \). Since \(|S^{(x)}| + |T^{(\beta)}| \leqslant 2(k - 1)m + 2 \), we get \( x + (mn - \beta + 1) \leqslant 2(k - 1)m + 2 \). Therefore, \( \beta - x \geqslant mn - 2(k - 1)m - 1 \geqslant (2kd + k - 1)m - 1 = (2kd - k + 1)m + 1 > (2d - 1)(km - 1) + 1 \). \( \square \)

By Claim 5, there are two integers \( x' \) and \( \beta' \) such that \( x' < x' < \beta' \leqslant \beta \) and \( \beta' - x' = (2d - 1)(km - 1) + 1 \). Let \( s = \beta' - (km - 1)d - 1 \), \( t = x' + (km - 1)d + 1 \), \( X = S^{(x')} \), and \( Y = T^{(\beta')} \). We remark that \(|X| + |Y| = (n - k)m + 1 \) and by \( \beta' - x' = (2d - 1)(km - 1) + 1 \geqslant (km - 1)d + 1 \), we get \( x' \leqslant s < t \leqslant \beta' \).

Claim 6. \(|X_i| + |Y_j| \leqslant m \) if \( v_i \in N_H[v_j] \).

Proof. For all \( x \in X \) and all \( y \in Y \), \( |f(x) - f(y)| \geqslant t-s = 2(km - 1)d - (\beta' - x') + 2 = km \).

Therefore, by Proposition 2 and \( B(G) \leqslant km - 1 \), we obtain \( km \leqslant (km - 1)d, x, y \). Hence, we get \( d_f(x, y) \geqslant 2 \). Thus, we get this claim. \( \square \)

Claim 7. (a) For every \( x \in S^{(x')} \) if \( x \in V(G_1) \), then \( Y_i = \emptyset \).

(b) For every \( y \in T^{(\beta')} \) if \( y \in V(G_2) \), then \( X_i = \emptyset \).
Proof. (a) By Proposition 2, we have \(|f(u) - f(x)| \leq B(I')d_I(u,x) \leq (km - 1)d\) for all \(u \in V(G_i)\), and hence \(f(u) \leq f(x) + (km - 1)d \leq x' + (km - 1)d < t\). Therefore, \(u \notin Y\), and this implies \(Y_i = \emptyset\).

(b) By Proposition 2, we obtain \(|f(v) - f(y)| \leq B(I')d_I(y,v) \leq (km - 1)d\) for all \(v \in V(G_j)\), and it follows that \(f(v) \geq f(y) - (km - 1)d \geq \beta' - (km - 1)d > s\). Therefore, \(v \in X\), and hence \(X_j = \emptyset\).  

By Claim 7 together with the fact that \(z \leq x' \leq s < t \leq \beta' \leq \beta\), we obtain \(|X_0|, |Y_0| \geq k\). Hence, by Claim 6 and Lemma 3, we get \(|X| + |Y| \leq (n - k)m\). On the other hand, \(|X| + |Y| = (n - k)m + 1\). This contradiction completes the proof of Theorem 1.

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References