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Note

Bandwidth of the composition of two graphs

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Abstract

The bandwidth $B(G)$ of a graph G is the minimum of the quantity $\max\{|f(x)-f(y)| : xy \in E(G)\}$ taken over all proper numberings f of G. The composition of two graphs G and H, written as $G[H]$, is the graph with vertex set $V(G)\times V(H)$ and with (u_1,v_1) is adjacent to (u_2,v_2) if either u_1 is adjacent to u_2 in G or $u_1=u_2$ and v_1 is adjacent to v_2 in H. In this paper, we investigate the bandwidth of the composition of two graphs. Let G be a connected graph. We denote the diameter of G by $D(G)$. For two distinct vertices $x,y\in V(G)$, we define $w_G(x,y)$ as the maximum number of internally vertex-disjoint (x,y)-paths whose lengths are the distance between x and y. We define $w(G)$ as the minimum of $w_G(x,y)$ over all pairs of vertices x,y of G with the distance between x and y is equal to $D(G)$. Let G be a non-complete connected graph and let H be any graph. Among other results, we prove that if $|V(G)|=B(G)D(G)-w(G)+2$, then $B(G[H])=(B(G)+1)|V(H)|-1$. Moreover, we show that this result determines the bandwidth of the composition of some classes of graphs composed with any graph.

Keywords: Bandwidth; Composition; Diameter; Density

1. Introduction

We consider finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For two vertices $x,y\in V(G)$, let $d_G(x,y)$ denote the distance between x and y in G, and let $D(G)$ denote the diameter of G. We write $\delta(G)$ and $\Delta(G)$ for the minimum degree and the maximum degree of a graph G, respectively. We denote the path, the cycle, and the complete graph on n vertices by P_n, C_n, and K_n, respectively. Let $K_{n_1,n_2,\ldots,n_k}$ denote the complete k-partite graph. We denote the kth power of a graph G by G^k.

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Let G be a graph on n vertices. A one-to-one mapping $f: V(G) \to \{1,2,\ldots,n\}$ is called a *proper numbering* of G. The *bandwidth of a proper numbering* f of G, denoted by $B_f(G)$, is the maximum difference between $f(x)$ and $f(y)$ when xy runs over all edges of G, namely,

$$B_f(G) = \max\{|f(x) - f(y)| : xy \in E(G)\}.$$

The *bandwidth* of G is defined to be the minimum of $B_f(G)$ over all proper numberings f of G, and denoted as $B(G)$, i.e.,

$$B(G) = \min\{B_f(G) : f \text{ is a proper numbering of } G\}.$$

A proper numbering f of G is called a *bandwidth numbering* of G when $B_f(G)=B(G)$.

The bandwidth problem for the graphs arises from sparse matrix computation, coding theory, and circuit layout of VLSI designs. Papadimitriou [12] proved that the problem of determining the bandwidth of a graph is NP-complete, and Garey et al. [5] showed that it remains NP-complete even if graphs are restricted to trees with maximum degree 3. Many studies have been done towards finding the bandwidth of specific classes of graphs (see [2,4,9]). In this paper, we investigate the bandwidth of the composition of two graphs.

The *composition* of two graphs G and H, written as $G[H]$, is the graph whose vertex set is $V(G) \times V(H)$ with two vertices (u_1, v_1) and (u_2, v_2) adjacent if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. There are some results on the bandwidth of the composition of specified graphs with any graph H. For instance, Chinn et al. [3] showed that $B(P_n^k[H])=(k+1)|V(H)|-1$ if $n \geqslant k+2$ and $B(C_n^k[H])=(2k+1)|V(H)|-1$ if $n \geqslant 2k+2$. Li and Lin [10] and Liu and Williams [11] established the bandwidths for $K_n[H]$ and $K_{1,n}[H]$. Moreover, Zhou and Yuan [15] gave the bandwidths of some composition graphs such as $(P_r \times P_s)[H]$, $(P_r \times C_s)[H]$ $(2r \neq s)$, $(C_r \times C_s)[H]$ $(6 \leqslant 2r \leqslant s)$, etc., where $G_1 \times G_2$ is the cartesian product of two graphs G_1 and G_2. The following upper bound for the bandwidth of the composition of two graphs is known.

Proposition 1 (Chinn et al. [2]). *For any two graphs G and H,*

$$B(G[H]) \leqslant (B(G) + 1)|V(H)| - 1.$$

Let G be a graph of order n. For $S \subseteq V(G)$, the neighborhood $N_G(S)$ is the set of all vertices v in $V(G) - S$ such that v is a vertex adjacent to at least one vertex in S. Let $\eta(G)$ denote $\max\min|N_G(S)|$, where the maximum is over all k with $1 \leqslant k \leqslant n$ and the minimum is over all $S \subseteq V(G)$ with $|S| = k$. We remark that $B(G) \geqslant \eta(G)$ [6]. The following lower bounds for the bandwidth of the composition of two graphs are known.

Proposition 2 (Chinn et al. [3]). *For any two graphs G and H,*

$$B(G[H]) \geqslant \eta(G)|V(H)| + \delta(H).$$

Proposition 3 (Li and Lin [10]). *For any two graphs G and H*,

$$B(G[H]) \geqslant |V(H)| + \left\lfloor \frac{\Delta(G)|V(H)| - 1}{2} \right\rfloor.$$

Moreover, Proposition 2 was generalized by Zhou and Yuan [15].

We study the bandwidth of the composition of two graphs, which satisfy an order condition. Let G be a connected graph. For two distinct vertices $x, y \in V(G)$, let $w_G(x, y)$ denote the maximum number of internally vertex-disjoint (x, y)-paths whose lengths are $d_G(x, y)$. We define $w(G)$ as the minimum of $w_G(x, y)$ taken over all pairs of vertices x, y of G satisfying $d_G(x, y) = D(G)$, i.e.,

$$w(G) = \min\{w_G(x, y) : x, y \in V(G) \text{ and } d_G(x, y) = D(G)\}.$$

For example, $w(T)=1$ if T is a tree, $w(C_{2n+1})=1$, $w(C_{2n})=2$, and $w(K_{m,n})=\min\{m, n\}$. Note that $w(G) \geqslant 1$ for any connected graph G. We get the following theorem, which gives a lower bound for the bandwidth of the composition of two graphs. To state our results we use $w(G)$.

Theorem 1. *Let k be a non-negative integer. Let G be a non-complete connected graph and let H be any graph. If $|V(G)| > (B(G)-k)D(G)-w(G)+2-D(G)/|V(H)|$, then*

$$B(G[H]) \geqslant (B(G) - k + 1)|V(H)| - 1.$$

From Theorem 1 together with Proposition 1, we have the following theorem.

Theorem 2. *Let G be a non-complete connected graph and let H be any graph. If $|V(G)| > B(G)D(G) - w(G) + 2 - D(G)/|V(H)|$, then*

$$B(G[H]) = (B(G) + 1)|V(H)| - 1.$$

Remark. For any connected graph G, $|V(G)| \leqslant B(G)D(G) - w(G) + 2$.

Furthermore, from Theorem 2, we obtain the following theorem, in which the condition has nothing to do with the structure of H.

Theorem 3. *Let G be a non-complete connected graph and let H be any graph. If $|V(G)| = B(G)D(G) - w(G) + 2$, then*

$$B(G[H]) = (B(G) + 1)|V(H)| - 1.$$

Theorem 3 determines the bandwidth of the composition of some classes of graphs with any graph. We consider the bandwidth of the composition of the complete bipartite graph $K_{m,n}$ ($\max\{m, n\} \geqslant 2$) with any graph H. Suppose that $m \geqslant n$. We verify that $|V(K_{m,n})| = m + n$, $D(K_{m,n}) = 2$, and $w(K_{m,n}) = n$, since $m \geqslant n$ and $\geqslant 2$. Moreover,

$B(K_{m,n}) = \lceil m/2 \rceil + n - 1$ when $m \geqslant n$ (see [2]). Therefore, if m is even, then $|V(K_{m,n})| = B(K_{m,n})D(K_{m,n}) - w(K_{m,n}) + 2$ $(m \geqslant n)$. Thus, by Theorem 3, we get the following corollary.

Corollary 1. *Let H be any graph. Let m and n be two positive integers satisfying $m \geqslant n$. If m is even, then*

$$B(K_{m,n}[H]) = (B(K_{m,n}) + 1)|V(H)| - 1 = \left(\frac{m}{2} + n\right)|V(H)| - 1.$$

However, if m is odd, then $B(K_{m,n})D(K_{m,n}) - w(K_{m,n}) + 2 = m + n + 1 > m + n = |V(K_{m,n})|$ $(m \geqslant n)$. Liu and Williams [11] proved that $B(K_{1,m}[H]) = \lfloor ((m+2)|V(H)| - 1)/2 \rfloor$ for any graph H. Hence, if m is odd and $|V(H)| \geqslant 2$, then we can verify that $B(K_{1,m}[H]) < (B(K_{1,m}) + 1)|V(H)| - 1$. This fact shows that the condition in Theorem 3 cannot be weakened.

This paper consists of three sections. In Section 2, we prove Theorem 1. In Section 3, we show the above Remark and some applications of Theorem 3.

2. Proof of Theorem 1

Let $m = |V(G)|$, $n = |V(H)|$, $b = B(G)$, $d = D(G)$, $w = w(G)$, and $\Gamma = G[H]$. Note that $d \geqslant 2$, since G is a non-complete connected graph. If $n = 1$, then the conclusion of Theorem 1 is true. So we may assume that $n \geqslant 2$. Write $V(G) = \{u_1, u_2, \ldots, u_m\}$. For $i = 1, 2, \ldots, m$, let H_i denote the graph induced by $\{u_i\} \times V(H)$ in Γ, i.e., $H_i = \Gamma[\{u_i\} \times V(H)]$. Let f be a bandwidth numbering of Γ. Write $x = f^{-1}(1)$ and $y = f^{-1}(mn)$. Without loss of generality, we may assume that $x \in V(H_1)$. By way of contradiction, assume that $B(\Gamma) \leqslant (b - k + 1)n - 2$.

Claim 1. *For all $z \in V(H_s)$ $(s \neq 1)$, $f(z) \leqslant d_G(u_1, u_s)(bn - kn - 1) + n$.*

Proof. Note that $d_G(u_1, u_s) \geqslant 1$, since $s \neq 1$. We prove this claim by induction on $d_G(u_1, u_s)$. Suppose that $d_G(u_1, u_s) = 1$. For all $z \in V(H_s)$, we have $|f(z) - f(x)| \leqslant B(\Gamma) \leqslant (b - k + 1)n - 2$, since $zx \in E(\Gamma)$. Therefore, we obtain $f(z) \leqslant (b - k + 1)n - 2 + f(x) = (bn - kn - 1) + n$ when $d_G(u_1, u_s) = 1$ and $z \in V(H_s)$. So we may assume that $d_G(u_1, u_s) \geqslant 2$. Let P be a shortest (u_1, u_s)-path in G, and let u_t be the vertex adjacent to u_s in P. We remark that $d_G(u_1, u_t) = d_G(u_1, u_s) - 1$ and $t \neq 1$, since $d_G(u_1, u_s) \geqslant 2$. By the induction hypothesis, we have $f(z') \leqslant d_G(u_1, u_t)(bn - kn - 1) + n = (d_G(u_1, u_s) - 1)(bn - kn - 1) + n$ for all $z' \in V(H_t)$. Hence by $|V(H_t)| = n$, there exists a vertex $v \in V(H_t)$ such that $f(v) \leqslant (d_G(u_1, u_s) - 1)(bn - kn - 1) + n - (n - 1) = (d_G(u_1, u_s) - 1)(bn - kn - 1) + 1$. Since $zv \in E(\Gamma)$ for all $z \in V(H_s)$, we get $|f(z) - f(v)| \leqslant B(\Gamma) \leqslant (b - k + 1)n - 2$, and it follows that $f(z) \leqslant f(v) + (b - k + 1)n - 2 \leqslant (d_G(u_1, u_s) - 1)(bn - kn - 1) + 1 + (b - k + 1)n - 2 = d_G(u_1, u_s)(bn - kn - 1) + n$ for all $z \in V(H_s)$. Thus, the conclusion of this claim is true. □

Claim 2. *Let v be a vertex of Γ. If $yv \in E(\Gamma)$, then $mn - (b - k + 1)n + 2 \leqslant f(v)$.*

Proof. By $yv \in E(\Gamma)$, we have $|f(y) - f(v)| \leqslant B(\Gamma) \leqslant (b-k+1)n-2$. Therefore, we get $mn-(b-k+1)n+2 \leqslant f(v)$, since $f(y)=mn$. □

Let $l = d_\Gamma(x,y)$. If $xy \in E(\Gamma)$, then $mn-1 = |f(x)-f(y)| \leqslant B(\Gamma) \leqslant (b-k+1)n-2 \leqslant mn-2$, since $b = B(G) \leqslant |V(G)|-1 = m-1$ and $k \geqslant 0$. From this contradiction and G is a non-complete connected graph, we may assume that $d \geqslant l \geqslant 2$.

Claim 3.

$$(bl-kl-m+2)n-l \geqslant \begin{cases} n & \textit{if } l<d, \\ nw & \textit{if } l=d. \end{cases}$$

Proof. We divide our proof into two cases.

Case 1: $y \in V(H_1)$. We remark that $l=2$, since $l \geqslant 2$ and G is a non-complete connected graph. Let $d_G(u_1)$ denote the degree of u_1 in G, and let $N_G(u_1)$ denote the neighborhood of u_1 in G. Note that $d_G(u_1) \geqslant 1$, since G is a non-complete connected graph. Moreover, note that if $d=2$, then $d_G(u_1) \geqslant \delta(G) \geqslant w$, where $\delta(G)$ is the minimum degree of G. We define $\tilde{H} = \Gamma[N_G(u_1) \times V(H)]$, i.e., $\tilde{H}$ is the graph induced by $N_G(u_1) \times V(H)$ in Γ. We remark that $xz, yz \in E(\Gamma)$ for all $z \in V(\tilde{H})$, since $x, y \in V(H_1)$. By $zx \in E(\Gamma)$ for all $z \in V(\tilde{H})$, we have $|f(z)-f(x)| \leqslant B(\Gamma) \leqslant (b-k+1)n-2$, and hence $f(z) \leqslant (b-k+1)n-2+f(x) = (b-k+1)n-1$. Therefore, from $|V(\tilde{H})| = nd_G(u_1)$, there exists a vertex $v \in V(\tilde{H})$ such that $f(v) \leqslant (b-k+1)n-1-(nd_G(u_1)-1) = (b-k+1)n-nd_G(u_1)$. Hence, by $yv \in E(\Gamma)$ and Claim 2, we get $mn-(b-k+1)n+2 \leqslant f(v) \leqslant (b-k+1)n-nd_G(u_1)$, and this implies that

$$(2b-2k-m+2)n-2 \geqslant nd_G(u_1) \geqslant \begin{cases} n & \text{if } d \geqslant 3, \\ nw & \text{if } d=2, \end{cases}$$

as desired, since $l=2$.

Case 2: $y \notin V(H_1)$. We argue as in Case 1. Without loss of generality, we may assume that $y \in V(H_2)$. Let $r = w_G(u_1,u_2)$. Note that $l = d_\Gamma(x,y) = d_G(u_1,u_2)$ and $r \geqslant 1$, since G is a connected graph. Furthermore, note that if $l=d$, then $r \geqslant w$. Let $P_1, P_2, \ldots, P_r$ be r internally vertex-disjoint (u_1,u_2)-paths whose lengths are l in G. For $i=1,2,\ldots,r$, let u_{a_i} be the vertex adjacent to u_2 in P_i. We remark that $d_G(u_1,u_{a_i}) = l-1 \geqslant 1$ and $u_2u_{a_i} \in E(G)$ for $i=1,2,\ldots,r$. By Claim 1, we obtain $f(z) \leqslant (l-1)(bn-kn-1)+n$ for all $z \in \bigcup_{i=1}^r V(H_{a_i})$. Hence, by $|\bigcup_{i=1}^r V(H_{a_i})| = nr$, there exists a vertex $v \in \bigcup_{i=1}^r V(H_{a_i})$ such that $f(v) \leqslant (l-1)(bn-kn-1)+n-(nr-1)$. Therefore, from $yv \in E(\Gamma)$ and Claim 2, we have $mn-(b-k+1)n+2 \leqslant f(v) \leqslant (l-1)(bn-kn-1)+n-nr+1$, and it follows that

$$(bl-kl-m+2)n-l \geqslant nr \geqslant \begin{cases} n & \text{if } l<d, \\ nw & \text{if } l=d. \end{cases}$$

Thus, we get this claim. □

Claim 4. $l=d$.

Proof. By way of contradiction, assume that $l < d$. By Claim 3 and $d-1 \geqslant l \geqslant 2$, we obtain

$$(bl - kl - m + 2)n - l \geqslant n$$

$$bln \geqslant (kl + m - 1)n + l$$

$$bln \cdot \frac{d}{ln} \geqslant ((kl + m - 1)n + l) \cdot \frac{d}{ln}$$

$$bd \geqslant \frac{d(m-1)}{l} + dk + \frac{d}{n}$$

$$\geqslant \frac{d(m-1)}{d-1} + dk + \frac{d}{n}.$$

We verify that $m \geqslant w(d-1)+2$, and it follows that $w \leqslant (m-2)/(d-1)$, since $d \geqslant 2$. By the assumption that $|V(G)| > (B(G)-k)D(G) - w(G) + 2 - D(G)/|V(H)|$, we have $bd < m+w+dk-2+d/n \leqslant m+(m-2)/(d-1)+dk-2+d/n = d(m-2)/(d-1)+dk+d/n$. Therefore, we get

$$\frac{d(m-1)}{d-1} + dk + \frac{d}{n} \leqslant bd < \frac{d(m-2)}{d-1} + dk + \frac{d}{n}$$

and this implies a contradiction, which completes the proof of the claim. □

We are now in a position to complete the proof of Theorem 1. By Claims 3 and 4, we obtain $(bd - kd - m + 2)n - d \geqslant nw$, and it follows that

$$mn \leqslant (b-k)dn - nw + 2n - d,$$

$$m \leqslant (b-k)d - w + 2 - \frac{d}{n},$$

which contradicts the assumption that $|V(G)| > (B(G)-k)D(G) - w(G) + 2 - D(G)/|V(H)|$. This contradiction completes the proof of Theorem 1.

3. Applications

Let G be a connected graph. The *density* of G is defined as $\lceil (|V(G)|-1)/D(G) \rceil$. We define the *local density* $\beta(G)$ of G to be the maximum density of all subgraphs of G, i.e., $\beta(G) = \max_{G' \subseteq G} \lceil (|V(G')|-1)/D(G') \rceil$. The following propositions are known.

Proposition 4 (Chinn et al. [2]). *Let G be a connected graph. Then,*

$$B(G) \geqslant \left\lceil \frac{|V(G)|-1}{D(G)} \right\rceil.$$

Proposition 5 (Chung [4]). *Let G be a connected graph. Then,*

$$B(G) \geqslant \beta(G).$$

Proposition 6 (Chinn et al. [2]). *Let G be a graph and let f be a bandwidth numbering of G. Then*

$$|f(x) - f(y)| \leqslant B(G)d_G(x, y) \quad \textit{for } x, y \in V(G).$$

From the Remark in Section 1, we can obtain an improvement of Propositions 4 and 5. First we show the following theorem, which improve Proposition 6.

Theorem 4. *Let G be a connected graph and let f be a bandwidth numbering of G. Then, for two distinct vertices $x, y \in V(G)$,*

$$|f(x) - f(y)| \leqslant \max\{B(G) - 1, B(G)d_G(x, y) - w_G(x, y) + 1\}.$$

Proof. If $xy \in E(G)$, then the conclusion of Theorem 4 is true, since $d_G(x, y) = w_G(x, y) = 1$ when $xy \in E(G)$. So we may assume that $d_G(x, y) \geqslant 2$. Let $r = w_G(x, y)$. Let $P_1, P_2, \ldots, P_r$ be r internally vertex-disjoint (x, y)-paths whose lengths are $d_G(x, y)$ in G. For $i=1, 2, \ldots, r$, let u_i be the vertex adjacent to y in P_i. Write $U=\{u_1, u_2, \ldots, u_r\}$. Note that $d_G(x, u) = d_G(x, y) - 1 \geqslant 1$ and $uy \in E(G)$ for all $u \in U$. Suppose that $|f(x)-f(u_1)|=|f(x)-f(u_2)|$. Then, we have $f(x)=(f(u_1)+f(u_2))/2$, and this implies that $|f(x)-f(y)|=|(f(u_1)+f(u_2))/2-f(y)|=\frac{1}{2}|f(u_1)+f(u_2)-2f(y)| \leqslant \frac{1}{2}(|f(u_1)-f(y)|+|f(u_2)-f(y)|) \leqslant \frac{1}{2}(2B(G)-1)$, since $u_1y, u_2y \in E(G)$ and $x \neq y$. Hence, we get $|f(x) - f(y)| \leqslant B(G) - 1$. Thus, we may assume that $|f(x) - f(u_i)| \neq |f(x) - f(u_j)|$ for $1 \leqslant i < j \leqslant r$. By Proposition 6, we obtain $|f(x)-f(u)| \leqslant B(G)d_G(x, u)= B(G)(d_G(x, y)-1)$ for all $u \in U$. Therefore, from $|U|=r=w_G(x, y)$, there exists a vertex $v \in U$ such that $|f(x)-f(v)| \leqslant B(G)(d_G(x, y)-1)-w_G(x, y)+1$. Since $vy \in E(G)$, we have $|f(v)-f(y)| \leqslant B(G)$. Hence, we get $|f(x)-f(y)| \leqslant |f(x)-f(v)|+|f(v)-f(y)| \leqslant B(G)(d_G(x, y)-1)-w_G(x, y)+1+B(G)=B(G)d_G(x, y)-w_G(x, y)+1$. □

Using Theorem 4, we can prove the following theorem, which is Remark in Section 1.

Theorem 5. *For any connected graph G,*

$$|V(G)| \leqslant B(G)D(G) - w(G) + 2.$$

Proof. Let f be a bandwidth numbering of G. Write $x=f^{-1}(1)$ and $y=f^{-1}(|V(G)|)$. If G is a complete graph, then the conclusion of this theorem is true. So we may assume that $D(G) \geqslant 2$. We divide our proof into two cases.

Case 1: $d_G(x, y) \leqslant D(G) - 1$. By Proposition 6 and $d_G(y,x) \leqslant D(G) - 1$, we have $|f(y) - f(x)| \leqslant B(G)d_G(y,x) \leqslant B(G)(D(G) - 1)$, and it follows that $|V(G)| \leqslant B(G)(D(G) - 1) + 1$ and $B(G) \geqslant (|V(G)| - 1)/(D(G) - 1)$. We remark that $|V(G)| \geqslant w(G)(D(G)-1)+2$, and this implies that $(|V(G)|-2)/(D(G)-1) \geqslant w(G)$. Therefore, we obtain $B(G) \geqslant (|V(G)|-1)/(D(G)-1) > (|V(G)|-2)/(D(G)-1) \geqslant w(G)$. Hence, we get $|V(G)| \leqslant B(G)(D(G) - 1) + 1 \leqslant B(G)D(G) - w(G) + 1 < B(G)D(G) - w(G) + 2$.

Case 2: $d_G(x, y) = D(G)$. Since G is a non-complete connected graph, we have $|f(x) - f(y)| = |V(G)| - 1 > B(G)$. Therefore, by Theorem 4, we obtain $|V(G)|-1=$

$|f(x)-f(y)| \leqslant B(G)d_G(x,y)-w_G(x,y)+1 \leqslant B(G)D(G)-w(G)+1$, and this implies that $|V(G)| \leqslant B(G)D(G)-w(G)+2$. □

From Theorem 5 (Remark in Section 1), we get the following corollary.

Corollary 2. *Let G be a connected graph. Then*,

$$B(G) \geqslant \left\lceil \frac{|V(G)|+w(G)-2}{D(G)} \right\rceil.$$

Since $w(G) \geqslant 1$ for any connected graph G, the lower bound in Corollary 2 is better than or equal to the density lower bound in Proposition 4. For example, we consider $B(P_3 \times P_3)$, where $G \times H$ is the cartesian product of two graphs G and H. By Proposition 4, we get $B(P_3 \times P_3) \geqslant \lceil (9-1)/4 \rceil = 2$. On the other hand, the lower bound in Corollary 2 gives $B(P_3 \times P_3) \geqslant \lceil (9+2-2)/4 \rceil = 3$. In fact, $B(P_3 \times P_3) = 3$.

By Corollary 2 and $B(G) \geqslant B(G')$ for $G' \subseteq G$, we obtain the following corollary, which is an improvement of Proposition 5.

Corollary 3. *Let G be a connected graph. Then*,

$$B(G) \geqslant \max_{G' \subseteq G} \left\lceil \frac{|V(G')|+w(G')-2}{D(G')} \right\rceil.$$

Next we show some applications of Theorem 3. Let n and k be two positive integers satisfying $n \geqslant k+2$. Let $a = \lfloor (n-2)/k \rfloor$ and $p = n-2-ka$, i.e., $n-2 = ka+p$ $(0 \leqslant p \leqslant k-1)$. We verify that $B(P_n^k) = k$, $D(P_n^k) = \lceil (n-1)/k \rceil = \lfloor (n+k-2)/k \rfloor = a+1$, and $w(P_n^k) = k-p$. Therefore, we get

$$\begin{aligned} B(P_n^k)D(P_n^k) - w(P_n^k) + 2 &= k(a+1) - (k-p) + 2 \\ &= (n-2-p+k) - (k-p) + 2 = n = |V(P_n^k)|. \end{aligned}$$

Hence, P_n^k $(n \geqslant k+2)$ satisfies the condition in Theorem 3. Similarly, we can show that if $n \geqslant 2k+2$, then $|V(C_n^k)| = B(C_n^k)D(C_n^k) - w(C_n^k) + 2$. Let n and k be two positive integers with $n \geqslant 2k+2$. Let $b = \lfloor (n-2)/2k \rfloor$ and $q = n-2-2kb$, namely, $n-2 = 2kb+q$ $(0 \leqslant q \leqslant 2k-1)$. We verify that $B(C_n^k) = 2k$, $D(C_n^k) = \lceil (n-1)/2k \rceil = \lfloor (n+2k-2)/2k \rfloor = b+1$, and $w(C_n^k) = 2k-q$. Therefore, we have

$$\begin{aligned} B(C_n^k)D(C_n^k) - w(C_n^k) + 2 &= 2k(b+1) - (2k-q) + 2 \\ &= (n-2-q+2k) - (2k-q) + 2 = n = |V(C_n^k)|. \end{aligned}$$

Moreover, we verify that for $m, n \geqslant 2$,

$$B(K_m \times P_n)D(K_m \times P_n) - w(K_m \times P_n) + 2 = mn - 2 + 2 = mn = |V(K_m \times P_n)|.$$

Thus, by Theorem 3, we obtain the following corollary, in which (i) and (ii) were proved by Chinn et al. [3].

Corollary 4. *Let H be any graph. Let m, n and k be positive integers.*

(i) $B(P_n^k[H]) = (B(P_n^k)+1)|V(H)| - 1 = (k+1)|V(H)| - 1$ *for* $n \geqslant k+2$ [3];
(ii) $B(C_n^k[H]) = (B(C_n^k)+1)|V(H)| - 1 = (2k+1)|V(H)| - 1$ *for* $n \geqslant 2k+2$ [3];
(iii) $B((K_m \times P_n)[H]) = (B(K_m \times P_n)+1)|V(H)| - 1 = (m+1)|V(H)| - 1$ *for* $m, n \geqslant 2$.

The *sum* (or *join*) of k graphs $G_1, G_2, \ldots, G_k$, denoted as $G_1 + G_2 + \cdots + G_k$, is the graph with vertex set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and edge set $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k) \cup \{uv : u \in V(G_i), v \in V(G_j) \text{ for } i \neq j\}$. Lai et al. [8] proved the following proposition.

Proposition 7 (Lai et al. [8]). *Let $k \geqslant 2$ be an integer. Let $G_1, G_2, \ldots, G_k$ be k graphs with $n_i = |V(G_i)|$ for $i = 1, 2, \ldots, k$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Write $n = \sum_{i=1}^k n_i$ and $G = G_1 + G_2 + \cdots + G_k$. If $B(G_1) < \lceil n_1/2 \rceil$, then*

$$B(G) = n - \left\lceil \frac{n_1+1}{2} \right\rceil = \left\lceil \frac{n_1}{2} \right\rceil + n_2 + \cdots + n_k - 1.$$

Furthermore, Proposition 7 was generalized by Li and Lin [10].

Proposition 8 (Li and Lin [10]). *Let $k \geqslant 2$ be an integer. Let $G_1, G_2, \ldots, G_k$ be k graphs with $n_i = |V(G_i)|$ for $i = 1, 2, \ldots, k$. Write $n = \sum_{i=1}^k n_i$ and $G = G_1 + G_2 + \cdots + G_k$. Then,*

$$B(G) = \min_{1 \leqslant i \leqslant k} \max \left\{ B(G_i) + n - n_i,\ n - \left\lceil \frac{n_i+1}{2} \right\rceil \right\}.$$

Let $G_1, G_2, \ldots, G_k$ be k graphs with $n_i = |V(G_i)|$ for $i = 1, 2, \ldots, k$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Let $G = G_1 + G_2 + \cdots + G_k$. Suppose that $n_1 \geqslant 2$ and $B(G_1) < \lceil n_1/2 \rceil$. Then G_1 is a non-complete graph, and it follows that $D(G) = 2$. We verify that $w(G) \geqslant n_2 + n_3 + \cdots + n_k$. Therefore, by Proposition 7, if n_1 is even, then

$$\begin{aligned} &B(G)D(G) - w(G) + 2 \\ &\quad \leqslant \left(\frac{n_1}{2} + n_2 + \cdots + n_k - 1 \right) 2 - (n_2 + n_3 + \cdots + n_k) + 2 \\ &\quad = n_1 + n_2 + \cdots + n_k = |V(G)|. \end{aligned}$$

Hence, from Theorems 3 and 5, we get the following proposition.

Proposition 9. *Let $k \geqslant 2$ be an integer. Let H be any graph. Let $G_1, G_2, \ldots, G_k$ be k graphs with $n_i = |V(G_i)|$ for $i = 1, 2, \ldots, k$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Write $G = G_1 + G_2 + \cdots + G_k$. If n_1 is even and $B(G_1) < n_1/2$, then*

$$B(G[H]) = (B(G)+1)|V(H)| - 1 = \left(\frac{n_1}{2} + n_2 + \cdots + n_k \right) |V(H)| - 1.$$

Proposition 9 implies the following corollary.

Corollary 5. *Let H be any graph. Let $k, n_1, n_2, \ldots, n_k$ be positive integers satisfying $k \geqslant 2$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. If n_1 is even, then*

$$B(K_{n_1,n_2,\ldots,n_k}[H]) = \left(\frac{n_1}{2} + n_2 + \cdots + n_k\right) |V(H)| - 1.$$

Let $k \geqslant 2$ and $h \geqslant 1$ be two integers. The *complete k-ary tree* of height h, denoted by $T_{k,h}$, has all its leaves (degree one vertices) at level h and all vertices at a level less than h have k children. Note that $|V(T_{k,h})| = 1 + k + k^2 + \cdots + k^h = (k^{h+1} - 1)/(k-1)$, $D(T_{k,h}) = 2h$, and $w(T_{k,h}) = 1$. Smithline [13] showed that the density lower bound in Proposition 4 determines the bandwidth of the complete k-ary trees.

Proposition 10 (Smithline [13]). *Let $k \geqslant 2$ and $h \geqslant 1$ be two integers. Then,*

$$B(T_{k,h}) = \left\lceil \frac{k(k^h - 1)}{2h(k-1)} \right\rceil.$$

By Theorem 3 and Proposition 10, we get the following corollary.

Corollary 6. *Let H be any graph. Let $k \geqslant 2$ and $h \geqslant 1$ be two integers. If $k + k^2 + \cdots + k^h \equiv 0 \pmod{2h}$, then*

$$B(T_{k,h}[H]) = (B(T_{k,h}) + 1)|V(H)| - 1 = \left(\frac{k(k^h - 1)}{2h(k-1)} + 1\right) |V(H)| - 1.$$

A *caterpillar* is a tree in which the removal of all vertices of degree one results a path. Sysło and Zak [14] proved that the local density lower bound in Proposition 5 is optimal for caterpillars.

Proposition 11 (Sysło and Zak [14]). *If G is a caterpillar, then $B(G) = \beta(G)$.*

A *k-caterpillar* is a tree formed from a path by growing edge-disjoint paths of lengths at most k from its vertices. We remark that a 1-caterpillar is a caterpillar. Proposition 11 was extended by Assmann et al. [1] to 2-caterpillars.

Proposition 12 (Assmann et al. [1]). *If G is a 2-caterpillar, then $B(G) = \beta(G)$.*

Furthermore, Hung et al. [7] extended Proposition 11 to a special class of block graphs which is called block caterpillar. A graph is a *block graph* if every block is a clique. A *block path* is a block graph with k cutvertices and $k+1$ blocks in which the cutvertices induce a path. A *block caterpillar* is a block graph in which deleting the vertices of degree one produces a block path. Note that 2-caterpillars are not generally block caterpillars.

Proposition 13 (Hung et al. [7]). *If G is a block caterpillar, then $B(G) = \beta(G)$.*

Let G be either a 2-caterpillar or a block caterpillar. From Propositions 12 and 13, there is a subgraph G' of G such that $B(G) = \lceil (|V(G')| - 1)/D(G') \rceil$. By $B(G) \geqslant B(G')$ and Proposition 4, we have $\lceil (|V(G')| - 1)/D(G') \rceil = B(G) \geqslant B(G') \geqslant \lceil (|V(G')| - 1)/D(G') \rceil$, and it follows that $B(G) = B(G') = \lceil (|V(G')| - 1)/D(G') \rceil$. Therefore, by Proposition 1 and Theorem 3, if $D(G') \geqslant 2$ and $|V(G')| \equiv 1 \pmod{D(G')}$, then $(B(G)+1)|V(H)| - 1 \geqslant B(G[H]) \geqslant B(G'[H]) = (B(G')+1)|V(H)| - 1 = (B(G)+1)|V(H)| - 1$ for any graph H, and hence we obtain $B(G[H]) = (B(G)+1)|V(H)| - 1$. Thus, we get the following corollary.

Corollary 7. *Let H be any graph. Let G be either a 2-caterpillar or a block caterpillar. Let G' be a subgraph of G such that $B(G) = \lceil (|V(G')| - 1)/D(G') \rceil$. If $D(G') \geqslant 2$ and $|V(G')| \equiv 1 \pmod{D(G')}$, then*

$$B(G[H]) = (B(G)+1)|V(H)| - 1.$$

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