JOURNAL OF COMBINATORIAL THEORY, Series B 62, 180–181 (1994)

Note

Every Planar Graph Is 5-Choosable

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Received August 24, 1993

We prove the statement of the title, which was conjectured in 1975 by V. G. Vizing and, independently, in 1979 by P. Erdös, A. L. Rubin, and H. Taylor.

A list coloring of a graph G is an assignment of colors to the vertices such that adjacent vertices get distinct colors and such that each vertex v receives a color in a prescribed list $L(v)$ of colors. G is *k-choosable* if such a coloring always exists provided that each $L(v)$ has k colors.

In 1975 Vizing raised the question whether every planar graph is 5-choosable (see [2]). Erdös *et al.* [1] conjectured that every planar graph is 5-choosable, but not necessarily 4-choosable. Recently, Voigt [3] described planar graphs that are not 4-choosable. In this paper we prove that they are always 5-choosable. The trick is to find an appropriate extension. The proof is probably the simplest proof of the 5-color theorem for planar graphs.

THEOREM. *Let G be a near-triangulation; i.e., G is a planar graph which has no loops or multiple edges and which consists of a cycle $C: v_1 v_2 \cdots v_p v_1$, and vertices and edges inside C such that each bounded face is bounded by a triangle. Assume that v_1 and v_2 are colored* 1 *and* 2, *respectively, and that $L(v)$ is a list of at least three colors if $v \in C - \{v_1, v_2\}$ and at least five colors if $v \in G - C$. Then the coloring of v_1 and v_2 can be extended to a list coloring of G.*

Proof (by induction on the number of vertices of G). If $p = 3$ and $G = C$ there is nothing to prove. So we proceed to the induction step.

If C has a chord $v_i v_j$, where $2 \leqslant i \leqslant j-2 \leqslant p-1$ $(v_{p+1} = v_1)$, then we apply the induction hypothesis to the cycle $v_1 v_2 \cdots v_i v_j v_{j+1} \cdots v_1$ and its interior and then to $v_j v_i v_{i+1} \cdots v_{j-1} v_j$ and its interior. So we can assume that C has no chord.

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Let $v_1, u_1, u_2, ..., u_m, v_{p-1}$ be the neighbors of v_p in that clockwise order around v_p. As the interior of C is triangulated, G contains the path $P: v_1 u_1 u_2 \cdots u_m v_{p-1}$. As C is chordless, $P \cup (C - v_p)$ is a cycle C'. Let x, y be two distinct colors in $L(v_p) \backslash \{1\}$. Now define $L'(u_i) = L(u_i) \backslash \{x, y\}$ for $1 \leqslant i \leqslant m$ and $L'(v) = L(v)$ if v is a vertex of G not in $\{u_1, u_2, ..., u_m\}$. Then we apply the induction hypothesis to C' and its interior and the new list L'. We complete the coloring by assigning x or y to v_p such that v_p and v_{p-1} get distinct colors. ▮

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