Note

Every toroidal graph without adjacent triangles is \((4, 1)^*\)-choosable

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Received 18 December 2003; received in revised form 4 December 2005; accepted 7 April 2006

Available online 5 September 2006

Abstract

In this paper, a structural theorem about toroidal graphs is given that strengthens a result of Borodin on plane graphs. As a consequence, it is proved that every toroidal graph without adjacent triangles is \((4, 1)^*\)-choosable. This result is best possible in the sense that \(K_7\) is a non-\((3, 1)^*\)-choosable toroidal graph. A linear time algorithm for producing such a coloring is presented also.

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MSC: 05C15; 05C78

Keywords: Triangle; Choosability; Toroidal graph; Linear algorithm

1. Introduction

All graphs considered are finite and simple. A torus is a closed surface (compact, connected 2-manifold without boundary) that is a sphere with a unique handle, and a toroidal graph is a graph embedable in the torus. For a toroidal graph \(G\), we still use \(G\) to denote an embedding of \(G\) in the torus.

Let \(G = (V, E, F)\) be a toroidal graph, where \(V, E\) and \(F\) denote the sets of vertices, edges and faces of \(G\), respectively. We use \(N_G(v)\) and \(d_G(v)\) to denote the set and number of vertices adjacent to a vertex \(v\), respectively, and use \(\delta(G)\) to denote the minimum degree of \(G\). A face of an embedded graph is said to be incident with all edges and vertices on its boundary. Two faces are adjacent if they share a common edge. The degree of a face \(f\) of \(G\), denoted also by \(d_G(f)\), is the length of the closed walk bounding \(f\) in \(G\). When no confusion may occur, we write \(N(v), d(v), d(f)\) instead of \(N_G(v), d_G(v), d_G(f)\). A \(k\)-vertex (or \(k\)-face) is a vertex (or face) of degree \(k\), a \(k^-\)-vertex (or \(k^-\)-face) is a vertex (or face) of degree at most \(k\), and a \(k^+\)-vertex (or \(k^+\)-face) is a vertex (or face) of degree at least \(k\). For \(f \in F(G)\), we write \(f = [u_1u_2 \ldots u_n]\) if \(u_1, u_2, \ldots, u_n\) are the vertices clockwise lying on the boundary of \(f\). An \(n\)-face \([u_1u_2u_3 \ldots u_n]\) is called an \((m_1, m_2, \ldots, m_n)\)-face if \(d(u_i) = m_i\) for \(i = 1, 2, \ldots, n\). An \(n\)-circuit is a circuit with exactly \(n\) edges.

In [7], Lebesgue proved a structural theorem about plane graphs that asserts that every 3-connected plane graph contains a vertex of given properties (see of [5, Theorem 2]). There are many analogous results appeared since then [1–3,5,10,14]. In this paper, we consider the structure of toroidal graphs, and prove a Lebesgue type theorem that strengthens a result given by Borodin in [2].

\(1\) Supported by NSFC 10371055.

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doi:10.1016/j.dam.2006.04.042
Theorem 1. Let $G$ be a connected toroidal graph. Then, one of the following holds:

1. $G$ contains two adjacent 3-faces.
2. $\delta(G) < 4$.
3. $G$ contains two adjacent 4-vertices.
4. $G$ contains a (4, 5, 5)-face.

A list assignment of $G$ is a function $L$ that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $L$-coloring with impropriety $d$ for integer $d \geq 0$, or simply an $(L, d)$-coloring, of $G$ is a mapping $\phi$ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that $\phi$ has at most $d$ neighbors colored with $\phi(v)$. For integers $m \geq d \geq 0$, a graph is called $(m, d)$-choosable, if $G$ admits an $(L, d)$-coloring for every list assignment $L$ with $|L(v)| = m$ for all $v \in V(G)$. An $(m, 0)$-choosable graph is simply called $m$-choosable.

The notion of list improper coloring was introduced independently by Škrekovski [11] and Eaton and Hull [4]. They proved that every plane graph without adjacent triangles is 4-choosable. This conjecture is still open.

The distances of two triangles $T_1$ and $T_2$ is defined to be the length of a shortest path connecting a vertex of $T_1$ to a vertex of $T_2$. Lam et al. [6] showed that every plane graph without triangles of distance less than 2 is $(4m, m)$-choosable. Xu [13,14] proved that every planar graph is $(3, 2)$-choosable and every outerplanar graph is $(2, 2)$-choosable. In [8], it was proved that every plane graph without 4-circuits and 5-circuits for some $l \in \{5, 6, 7\}$ is $(3, 1)$-choosable.

Theorem 2. Let $G$ be a toroidal graph without adjacent triangles. Then $G$ is $(4, 1)$-choosable.

Since $K_7$ is a toroidal graph, and it is not $(L, 1)$-choosable for $L(v) = \{1, 2, 3\}$ for each of its vertices $v$, Theorem 2 is best possible in this sense.

In Section 2, we give the proofs of our theorems. According to the proof of Theorem 2, a linear time algorithm is given in Section 3.

2. Proofs of the theorems

Proof of Theorem 1. Assume to the contrary that the theorem is false. Let $G$ be a connected toroidal graph with the properties that $G$ contains no adjacent 3-faces, $\delta(G) \geq 4$, every 4-vertex is adjacent to only $5^+$-vertices, and every 3-face is not a (4, 5, 5)-face. The Euler’s formula $|V| + |F| - |E| \geq 0$ can be rewritten in the following form:

$$\sum_{v \in V(G)} \left\{ \frac{3 \cdot d_G(v)}{10} - 1 \right\} + \sum_{f \in F(G)} \left\{ \frac{d_G(f)}{5} - 1 \right\} \leq 0. \hspace{1cm} (1)$$

Let $\omega$ be a weight on $V(G) \cup F(G)$ by defining $\omega(v) = (3 \cdot d(v)/10) - 1$ if $v \in V(G)$, and $\omega(f) = (d(f)/5) - 1$ if $f \in F(G)$. Then the total sum of the weights is no more than zero. To prove Theorem 1, we will introduce some rules to transfer weights between the elements of $V(G) \cup F(G)$ so that the total sum of the weights is kept constant while the transferring is in progress. However, once the transferring is finished, we can show that the resulting weight $\omega'$ satisfying $\sum_{x \in V(G) \cup F(G)} \omega'(v) > 0$. This contradiction to (1) will complete the proof.

Our transferring rules are as follows:

1. A 4-vertex transfers $\frac{1}{20}$ to each incident 3-face or 4-face.
2. A $5^+$-vertex transfers $\frac{7}{20}$ to each incident 3-face.
3. A 5-vertex transfers $\frac{1}{20}$ to each incident 4-face.
4. A $6^+$-vertex transfers $\frac{11}{20}$ to each incident 4-face.
Let \( v \) be a \( k \)-vertex of \( G \). Since \( G \) contains no adjacent 3-faces, \( v \) is incident with at most \( \lfloor k/2 \rfloor \) 3-faces. If \( k = 4 \), then by (R1), \( \omega'(v) \geq \omega(v) - \frac{1}{20} \cdot 4 = 0 \). If \( k = 5 \) and there is no 3-face incident with \( v \), then by (R3),
\[
\omega'(v) \geq \omega(v) - \frac{1}{20} \cdot 5 = \frac{10 - 5}{20} > 0.
\] (2)
If \( k = 5 \) and there are 3-faces incident with \( v \), then the number of 3-faces incident with \( v \) is at most two, and thus
\[
\omega'(v) \geq \omega(v) - \frac{7}{20} \cdot 2 - \frac{3}{20} \cdot 3 = 0 \quad \text{by (R2) and (R3).}
\]
If \( k \geq 6 \) and there is no 3-face incident with \( v \), then by (R4),
\[
\omega'(v) \geq \omega(v) - \frac{11}{120} \cdot k = (25k - 120)/120 > 0.
\]
If \( k \geq 6 \) and there are 3-faces incident with \( v \), then
\[
\omega'(v) \geq \omega(v) - \frac{7}{40} \cdot \left( \left\lfloor \frac{k}{2} \right\rfloor - \frac{11}{120} \cdot \left( k - \left\lfloor \frac{k}{2} \right\rfloor \right) \right) \geq \frac{20k - 120}{120} \geq 0.
\] (3)

Let \( f \) be an \( h \)-face of \( G \). If \( h = 3 \), then by our choice of \( G \), either \( f \) is incident with three \( 5^+ \)-vertices and thus by (R2)
\[
\omega'(f) = \omega(f) + \frac{7}{40} \cdot 3 > 0
\] (4)
or \( f \) is incident with a unique 4-vertex and at least one \( 6^+ \)-vertex and thus by (R1) and (R2),
\[
\omega'(f) = \omega(f) + \frac{1}{20} + \frac{7}{40} \cdot 2 = 0.
\]
If \( h = 4 \), since \( f \) is incident with four \( 4^+ \)-vertices, then \( \omega'(f) \geq \omega(f) + \frac{1}{20} \cdot 4 = 0 \) while \( f \) is incident with no \( 6^+ \)-vertex, and
\[
\omega'(f) = \omega(f) + \frac{1}{20} \cdot 3 + \frac{11}{120} = \frac{1}{24} > 0 \quad \text{while } f \text{ is incident with a } 6^+\text{-vertex.}
\] (5)
If \( h \geq 5 \), then \( \omega'(f) = \omega(f) > 0 \).
Now, we get that \( \omega'(x) \geq 0 \) for each \( x \in V(G) \cup F(G) \). It follows that \( 0 \leq \sum_{x \in V(G) \cup F(G)} \omega'(x) = \sum_{x \in V(G) \cup F(G)} \omega(x) \leq 0 \).
If \( \sum_{x \in V(G) \cup F(G)} \omega'(x) > 0 \), we are done. Assume that \( \sum_{x \in V(G) \cup F(G)} \omega'(x) = 0 \). Then, by (3), \( G \) contains no \( 7^+ \)-vertices, and every \( 6^+ \)-vertex is incident with three \( 3^+ \)-faces and three \( 4^+ \)-faces, but this implies that \( \omega'(f') > 0 \) for every \( 4^+ \)-face \( f' \) incident with this \( 6^+ \)-vertex by (5). Therefore, we may assume that \( G \) contains no \( 6^+ \)-vertices, and hence every \( 3^+ \)-face \( f'' \) is incident with three \( 5^+ \)-vertices that indicates \( \omega(f'') > 0 \) by (4). So, \( G \) contains no 3-faces. But this indicates that \( G \) contains no vertices of degree 5 by (2), and hence \( G \) is 4-regular. This contradicts to the choice of \( G \), and ends the proof. \( \square \)

**Proof of Theorem 2.** Assume to the contrary. Let \( G \) be a counterexample with the fewest vertices, i.e., \( G \) is a non-\( (4, 1)^* \)-choosable toroidal graph without adjacent triangles, but any proper subgraph of \( G \) is \( (4, 1)^* \)-choosable. It is certain that we may assume that \( G \) is connected.

Let \( L \) be a list assignment of \( G \) satisfying \( |L(v)| = 4 \) for all \( v \in V(G) \) such that \( G \) is not \( (L, 1)^* \)-choosable.
We will show that \( \delta(G) \geq 4 \), and \( G \) contains neither two adjacent 4-vertices nor a \((4,5,5)\)-face. This contradiction to Theorem 1 will complete our proof.
If \( \delta(G) < 4 \), let \( v \) be a \( 3^- \)-vertex of \( G \). Then, \( G - v \) is \( (4, 1)^* \)-choosable by the choice of \( G \). Since in any \( (L, 1)^* \)-coloring of \( G - v \), there must exist a color in \( L(v) \) that is not used by any neighbors of \( v \), any \( (L, 1)^* \)-coloring of \( G - v \) can be extended to an \( (L, 1)^* \)-coloring of \( G \), a contradiction.
If \( G \) contains two adjacent 4-vertices, say \( u \) and \( v \), then by the choice of \( G \), \( G - \{u, v\} \) is \( (4, 1)^* \)-choosable. By the same argument as above, we get an \((L, 1)^* \)-coloring of \( G \), a contradiction also.
If \( G \) contains a \((4, 5, 5)\)-face \( f = [xyz] \), we may assume that \( d(x) = 4 \) and \( d(y) = d(z) = 5 \). Let \( H = G - \{x, y, z\} \).
By the choice of \( G \), \( H \) admits an \((L, 1)^* \)-coloring \( \phi \). For \( w \in \{x, y, z\} \), let \( L'(w) = L(w) \setminus \{\phi(u) | u \in N_H(w)\} \). Then, \( |L'(x)| \geq 2 \), \( |L'(y)| \geq 1 \) and \( |L'(z)| \geq 1 \). If \( L'(y) = L'(z) \), then color \( y \) and \( z \) with the same color \( \gamma \) in \( L'(y) \) and color \( x \) with a color in \( L'(x) \setminus \{\gamma\} \). If \( L'(y) \neq L'(z) \), then color \( y \) with a color \( x \in L'(y) \setminus L'(z) \), color \( z \) with a color in \( L'(z) \), and color \( x \) with an arbitrary color in \( L'(x) \). In either case, we get an \((L, 1)^* \)-coloring of \( G \). This contradiction completes the proof of Theorem 2. \( \square \)
3. A linear time algorithm

In [9], Mohar presented a linear time algorithm that for every fixed surface $S$ and a given graph $G$, either finds an embedding of $G$ in $S$ or returns a subgraph of $G$ that is a subdivision of a Kuratowski graph for $S$.

From the proof of Theorem 2, we give here a linear time algorithm that, for an arbitrary toroidal graph $G$ without adjacent triangles, produces an $(L, 1)^*$-coloring for any fixed list assignment $L$ with $|L(v)| = 4$ for each $v \in V(G)$.

The strategy of our algorithm is as follows. First, we repeatedly locate a $3^-$-vertex, or a pair of adjacent $4$-vertices, or three vertices incident with a $(4, 5, 5)$-face until no $3^+$-vertices remain. At the end of the above process, what remains is a subgraph of maximum degree at most 2, say $H$. Then, we color $H$ with the given color lists greedily, and extend the coloring step by step to whole $G$ as shown in the proof of Theorem 2.

$(4, 1)^*$.Coloring Toroidal Graphs without Adjacent Triangles

**Input:** An embedding of a connected toroidal graph $G$ without adjacent triangles, and a list assignment $L$ with $|L(v)| \geq 4$ for each $v \in V(G)$.

**Output:** An $(L, 1)^*$-coloring $\phi$ of $G$.

**Step 0:** Set $i = 0$, $G_0 = G$, $V_0 = \{v|d(v) \leq 3\}$, $E_0 = \{uv|u, v \notin V_0$ and $d(u) = d(v) = 4\}$, and $F_0 = \{f = [uvw]\}$ $[u, v, w] \notin V_0 \cup V(E_0)$ and $d(u) + d(v) + d(w) = 14$.

**Step 1:** If $\delta(G_i) \leq 2$, color $G_i$ with a proper coloring $\phi$ greedily, and goto Step 3.

**Step 2:** If $V_0 \neq \emptyset$, choose $v \in V_0$, set $S_i := \{v\}$ and reset $V_0 := V_0 \setminus \{v\}$;

**else if** $E_0 \neq \emptyset$, choose $uv \in E_0$, set $S_i := \{u, v\}$ and reset $E_0 := E_0 \setminus \{uv\}$;

**else choose** an $f = [uvw] \in F_0$, set $S_i := \{u, v, w\}$ and reset $F_0 := F_0 \setminus \{f\}$.

 Reset $G_i := G_i - S_i$, $i := i + 1$, and add the new $3^+$-vertices, adjacent $4$-vertices, and $(4, 5, 5)$-face of $G_i$ into $V_0$, $E_0$ and $F_0$, respectively. Goto Step 1.

**Step 3:** If $i = 0$, output $\phi$.

**Step 4:** If $S_{i-1} = \{u\}$, color $u$ by $\phi(u) \in L(u)\setminus \{\phi(x)|x \in V(G_i), ux \in E(G)\}$;

**else if** $S_{i-1} = \{u, v\}$, color $u$ by $\phi(u) \in L(u)\setminus \{\phi(x)|x \in V(G_i), ux \in E(G)\}$, and color $v$ by $\phi(v) \in L(u)\setminus \{\phi(x)|x \in V(G_i), ux \in E(G)\}$;

**else** $S_{i-1} = \{u, v, w\}$, choose $\phi(u)$, $\phi(v)$ and $\phi(w)$ as described in the proof of Theorem 2 for $u$, $v$ and $w$, respectively.

 Reset $i := i - 1$ and goto Step 3.

 From the proof of Theorem 2, one can easily verify that this algorithm works correctly. Now, we analyze its complexity. Given an embedding of a toroidal graph without adjacent triangles, it takes at most $O(n)$ time to produce $V_0$, $E_0$ and $F_0$ in Step 0. Each time Step 2 is executed, the vertices in $S_i$ are removed, and $V_0$, $E_0$ and $F_0$ can be modified in constant time after removing the vertices in $S_i$. So, it totally takes linear time to run Step 2. After that the algorithm takes another $O(n)$ time to run Step 4 for finding a color for every vertex. Thus, this algorithm is a linear time algorithm.

 Combined with Mohar’s algorithm for finding an embedding of a toroidal graph in the torus, for any given graph $G$, we can, in linear time, either find an $(L, 1)^*$-coloring for an arbitrary list assignment $L$ with $|L(v)| = 4$ for every $v \in V(G)$ or conclude that $G$ is either a non-toroidal graph or a toroidal graph that contains adjacent triangles.

 **Conclusion:** Although we have an example $K_7$ that is non-(4, 1)*-choosable toroidal graph, but $K_7$ has too much triangles compared with its order. It seems that the condition “without adjacent triangles” is far away from a tight condition for (4, 1)*-choosable plane graphs. Finally, we propose a question analogue to the conjecture of Lam et al.

 **Question:** Is it true that every plane graph without adjacent triangles is (3, 1)*-choosable?

Acknowledgements

The authors appreciate the referees sincerely for their helpful comments.

References