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Information Processing Letters 54 (1995) 301-304

Information Processing Letters

On the equivalence covering number of splitgraphs

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Communicated by P.M.B. Vitányi; received 8 September 1994

Abstract

An equivalence graph is a disjoint union of cliques. For a graph G let $eq(G)$ be the minimum number of equivalence subgraphs of G needed to cover all edges of G. We call $eq(G)$ the equivalence covering number of G. It was shown in [8] that computing the equivalence covering number is NP-hard, even when restricted to graphs in which no two triangles have a vertex in common. We show that the equivalence covering number for splitgraphs can be approximated within an additive constant 1. We also show that obtaining the exact value of the equivalence number of a splitgraph is an NP-hard problem. Using a similar method we also show that it is NP-complete to decide whether the equivalence covering number of a graph is 3, even for graphs with maximum degree 6 and with maximum clique number 4.

Keywords: Algorithms; Graph algorithms; Equivalence covering; Splitgraphs

1. Introduction

Definition 1. An *equivalence graph* is a vertex disjoint union of cliques. An *equivalence covering* of a graph G is a family of equivalence subgraphs of G such that every edge of G is an edge of at least one member of the family. The *equivalence covering number* of G, denoted by $eq(G)$, is the minimum cardinality of all equivalence coverings of G.

The equivalence covering number was studied first in [2]. Interesting bounds for the equivalence covering number in terms of maximal degree of the complement were obtained in [1]. In this note we mainly consider the computation of the equivalence covering number of splitgraphs. We first show an approximation within an additive constant 1. Then we show that obtaining the exact value is an NP-hard problem.

Definition 2. A graph $G = (V, E)$ is a *split graph*, if there is a partition $V = S + K$ of its vertex set into a stable set S and a clique K.

There is no restriction on edges between vertices of S and vertices of K. Notice that in general the partition into S and K need not be unique. Splitgraphs are exactly those graphs which, together with their complements, are chordal. For more general information on splitgraphs we refer to [4].

2. Approximation

In this section we show that the equivalence covering number of a splitgraph can be approximated within an additive constant 1. Consider a partition $V = S + K$

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0020-0190/95/$09.50
SSDI 0020-0190(95)00037-2

of the vertex set into an independent set S and a clique K. For a vertex x in K let $\delta(x)$ be the number of neighbors of x in S. Let $\Delta = \max\{\delta(x) \mid x \in K\}$.

Lemma 3. $eq(G) \geqslant \Delta$.

Proof. Consider a vertex $x \in K$ with $\delta(x) = \Delta$ and its neighbors in S. This is a $K_{1,\Delta}$ induced subgraph of G. This induced subgraph has equivalence covering number Δ, since each equivalence graph in the covering can have only one edge. This proves the lemma. □

Lemma 4. $eq(G) \leqslant \Delta + 1$.

Proof. Let $y_1, \ldots, y_t$ be the vertices of S. For each vertex x in K consider an arbitrary ordering of its neighbors in S. For $i = 1, \ldots, \Delta$ define the equivalence graph G_i as follows. G_i is the disjoint union of cliques $W_{i,j} = \{y_j\} \cup \{x \in K \mid \text{the } i\text{th neighbor of } x \text{ is } y_j\}$, for $j = 1, \ldots, t$. It is easy to check that the cliques $W_{i,j}$ for $j = 1, \ldots, t$ are all disjoint. We define one more equivalence graph $G_{\Delta+1}$ consisting of the clique K. Obviously, this gives an equivalence covering with $\Delta + 1$ equivalence graphs. □

The approximation given in Lemma 4 can be computed in linear time. This proves the following theorem.

Theorem 5. *There exists a linear time algorithm to compute an equivalence covering of a splitgraph G with at most $eq(G) + 1$ equivalence graphs.*

Remark 6. Notice that, in case the splitgraph is a threshold graph (see, e.g., [4]), its equivalence number can easily be computed exactly.

3. NP-completeness

We use a reduction from EDGE-COLORING.

The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the number of colors required to color the edges of the graph in such a way that no two adjacent edges have the same color. By Vizing's theorem (see, e.g., [3]) the chromatic index is either d or $d + 1$, where d is the maximum vertex degree.

Notice that, in general, the chromatic index is an upperbound for the equivalence covering number. Also, these parameters coincide for triangle-free graphs. It follows that, for bipartite graphs, the equivalence covering number equals the maximum degree. Unfortunately, for splitgraphs the bound is not of much use, which is illustrated by a clique.

It is by now well known that it is NP-complete to determine the chromatic index of an arbitrary graph [5,6]. Holyer [5] obtained the following result.

Theorem 7. *It is* NP-*complete to determine whether the chromatic index of a cubic graph is* 3 *or* 4.

Consider a cubic graph G and construct a graph H as follows. For each edge e of G introduce a new vertex x_e and make this adjacent to the two endvertices of e. We call x_e the *special* vertex at e.

Lemma 8. $\chi'(G) = 3 \Leftrightarrow eq(H) = 3$.

Proof. First assume $\chi'(G) = 3$. Notice that $eq(H) \geqslant 3$ since H has an induced $K_{1,3}$ subgraph. (If p is a vertex of G incident with edges e, f and g in G, then $\{p, x_e, x_f, x_g\}$ induces a $K_{1,3}$ in H.) Consider an edge coloring of G with three colors. For each color class define an equivalence graph as follows. For each edge in that color class, the triangle consisting of the edge and the special vertex at that edge is a clique of the equivalence graph. It is easy to check that this defines an equivalence covering with three equivalence graphs.

Now assume H has an equivalence covering with three equivalence graphs H_1, H_2 and H_3. We claim that no triangle of G is contained in a clique of one of the equivalence graphs. Assume, by way of contradiction, that $\{a, b, c\}$ is a triangle of G which is contained in a clique of H_1. Vertex a is adjacent to three special vertices, say x_1, x_2 and x_3. Then each of the edges (a, x_i) is contained in a clique of an equivalence graph, and no two are in a clique of the same equivalence graph. Without loss of generality we may assume that (a, x_i) is contained in a clique of H_i. But then H_1 cannot contain the triangle $\{a, b, c\}$ since x_1 has degree two and hence the clique containing a and x_1 can have at most two vertices of G.

We can color the edges of G as follows. If the edge e is contained in a clique of H_i then we give it color i. (If e is contained in cliques of more than one equivalence graph, we can choose one arbitrarily). By the remark above this gives a correct edge-coloring with three colors. □

Corollary 9. *It is* NP-*complete to decide whether the equivalence covering number of a graph with maximum degree* $\leqslant 6$ *and without a copy of* K_4 *equals* 3.

Given a cubic graph G we construct a splitgraph G^* as follows. The vertex set of G^* is split into a clique K and an independent set S. The vertices of K are the vertices of G. For each edge e of G introduce two new vertices $x_{e,1}$ and $x_{e,2}$ which are both made adjacent to the endvertices of e. For each nonedge f of G, we introduce *one* new vertex y_f which is made adjacent to the endvertices of f. We again call the new vertices, which are the vertices of S, special vertices.

Lemma 10. $\chi'(G) = 3 \Leftrightarrow eq(G^*) = n + 2$, *where* n *is the number of vertices of* G.

Proof. The proof goes along the same lines as the proof of Lemma 8. Assume G can be edge-colored with three colors. Notice that $eq(G^*) \geqslant n + 2$ since $K_{1,n+2}$ is an induced subgraph. Since G is cubic, n is even. We can construct an equivalence covering for G^* as follows. First, consider an edge-coloring of K with $n - 1$ colors (see [3]). For each color class, define an equivalence graph as follows. For each edge in K in that color class, add one special vertex at that edge and let that triangle be a clique of the equivalence graph.

Next consider an edge-coloring of G with three colors. For each color class define an equivalence graph as follows. For each edge in that color class add the other special vertex and let that triangle be a clique of the equivalence graph.

Clearly, this defines an equivalence covering of G^* with $n + 2$ equivalence graphs.

Assume that G^* has an equivalence covering with $n + 2$ equivalence graphs. Consider a vertex $a \in K$. This vertex a is adjacent to $n + 2$ special vertices, and each of the edges between a and a special vertex defines a unique equivalence graph. It follows that no triangle of G can be contained in a clique of an equivalence graph. We thus obtain a correct edge-coloring of G in the same manner as in the proof of Lemma 8. □

Corollary 11. *It is* NP-*complete to decide whether the equivalence covering number of a splitgraph, in which every vertex of the independent set has degree two, is* Δ *or* $\Delta + 1$, *where* $\Delta = \max\{\delta(x) \mid x \in K\}$ *for a given partition of the vertex set into a clique* K *and an independent set* S.

4. Concluding remarks

In this note we considered the equivalence covering number for splitgraphs. We show that it is NP-complete to determine the equivalence covering number of a splitgraph. Another complexity result was obtained in [8]. In this paper the authors show that the equivalence covering number is NP-complete even for graphs in which no two triangles have a vertex in common.

Related problems are the clique covering number, and the clique partition number. The clique covering number is the minimum number of cliques which cover all the edges of the graph. It was shown in [9] that the clique covering number can be computed in linear time for chordal graphs. The clique partition number is the minimum number of cliques such that every edge is contained in *exactly* one clique. Determining the clique partition number is NP-hard for chordal graphs [10]. It would be interesting to determine the complexity of the computation of the clique partition number for split graphs. It should be remarked however that it is unlikely that a polynomial time algorithm exists, due to the following [11,7]. Consider the following splitgraph G. Take a clique with $m^2 + m + 1 - r$ vertices and an independent set with r vertices. Make every vertex of the independent set adjacent to every vertex of the clique. (G is sometimes denoted as $K_{m^2+m+1} \setminus K_r$.) If $2 < r < m^2+m+1$ then the clique partition number of G is at least m^2+m with equality holding if and only if a projective plane of order m exists and $r = m + 1$.

Acknowledgements

We thank an anonymous referee for drawing our attention to [8].

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