On set intersection representations of graphs

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Abstract

The intersection dimension of a bipartite graph with respect to a type $L$ is the smallest number $t$ for which it is possible to assign sets $A_x \subseteq \{1, \ldots, t\}$ of labels to vertices $x$ so that any two vertices $x$ and $y$ from different parts are adjacent if and only if $|A_x \cap A_y| \in L$. The weight of such a representation is the sum $\sum_x |A_x|$ over all vertices $x$. We exhibit explicit bipartite $n \times n$ graphs whose intersection dimension is: (i) at least $n^{1/|L|}$ with respect to any type $L$, (ii) at least $\sqrt{n}$ with respect to any type of the form $L = \{k, k+1, \ldots\}$, and (iii) at least $n^{1/|R|}$ with respect to any type of the form $L = \{k : k \mod p \in R\}$, where $p$ is a prime number. We also show that any intersection representation of a Hadamard graph must have weight about $n \log n$, independent on the used type $L$. Finally, we formulate several problems about intersection dimensions of graphs related to some open problems in the complexity of boolean functions.

Key words and phrases: Intersection graphs, clique covering number, bicliques, Hadamard graphs, Sylvester graphs, Ramsey graphs, boolean functions

1 Introduction

We consider representations of graphs as intersection graphs of families of sets. The size of the underlying set serves as a measure of complexity. Various conditions on when we draw an edge between the sets give various measures. Our motivation is that, for bipartite graphs, these measures capture the computational complexity of boolean functions (see Section 8).

A general scenario is the following. Given a graph $G$, we want to assign finite sets $A_x$ of labels to its vertices $x$ so that two vertices are adjacent iff $|A_x \cap A_y| \in L$. Here $L$ is the type of the representation, and the total number of used labels is its size. Given a type $L \subseteq \{0, 1, \ldots\}$, the goal is to minimize the number of used labels. That is, we are interested in the smallest number $\Theta_L(G)$ of labels needed to represent the graph with respect to the type $L$.

When dealing with intersection representations of a graph, it is often useful to keep in mind that this is equivalent to covering its edges by complete graphs. Namely, $\Theta_L(G)$ is the smallest

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number $t$ for which there exist $t$ subsets $S_1, \ldots, S_t$ of vertices such that two vertices $x$ and $y$ are adjacent in $G$ iff the number of $S_i$’s containing both $x$ and $y$ belongs to $L$. To see this connection, just associate with each vertex $x$ the set of labels $A_x = \{i : x \in S_i\}$.

Different types $L$ lead to different measures $\theta_L(G)$. The following types have recently drawn considerable attention. The threshold-$k$ dimension $\Theta_k(G)$ is the intersection dimension with respect to the type $L = \{k, k+1, \ldots\}$. The intersection rule in this case is $xy \in E$ iff $|A_x \cap A_y| \geq k$. The threshold dimension is the minimum $\Theta_{\text{thd}}(G) = \min_k \Theta_k(G)$ over all threshold values $k$. The next important class of types are modular types of the form $L = \{k : k \mod p \in R\}$. In this case the intersection rule is $xy \in E$ iff $|A_x \cap A_y| \mod p \in R$. The minimum $\Theta_{\text{mod}}(G) = \min_L \Theta_L(G)$ over all modular types $L$ is the modular dimension of $G$. The simplest of these measures is the parity dimension $\Theta_{\text{odd}}(G)$ which corresponds to the type $L = \{k : k \mod 2 = 1\}$. The most difficult measure to deal with is the absolute dimension $\Theta(G) = \min_L \Theta_L(G)$, where the minimum is over all types $L$.

Among all these measures the best understood is the threshold-1 dimension $\Theta_1(G)$. In this case the intersection rule is $xy \in E$ iff $A_x \cap A_y \neq \emptyset$. This measure is also known as the edge clique covering number, that is, the smallest number of complete subgraphs of a graph covering all its edges. For this measure the following bounds are known.\footnote{Since we are only interested in the rate of growth of the corresponding measures, we will ignore absolute multiplicative factors and write $f(n) \leq g(n)$ if $f(n) = O(g(n))$, that is, if there is an absolute constant $c > 0$ such that $f(n) \leq cg(n)$ for all $n$. All logarithms are to the base of 2.}

- The classical result of Erdős, Goodman and Pósa [14] states that $\Theta_1(G) \leq \Theta_1(K_{n,n}) = n^2$ for every graph $G$ on $2n$ vertices.

- Results of Frieze and Reed [18], and Eaton and Grable [10] imply that, for every fixed $k$, random graphs on $n$ vertices have somewhat smaller threshold-$k$ dimension $\Theta_k(G)$, which is about $n^2/\log^2 n$.

- A general upper bound $\Theta_1(G) \leq \Delta^2 \log n$ for all $n$-vertex graphs $G$ of maximum degree $\Delta$ was proved by Alon [2].

- A tighter upper bound $\Theta_1(G) \leq \delta^2 \log n$, where $\delta$ is the maximum of the average degrees of induced subgraphs of $G$, was proved by Eaton and Rödl [9].

- They also proved that $n$-vertex graphs $G$ of maximum degree $\Delta$ with $\Theta_1(G) \geq \frac{\Delta^2}{\log \Delta} \log \frac{n}{\Delta}$ exist.

Note that we cannot expect similar upper bounds for the graph $G$ itself, just because $\Theta_1(G)$ is equal to the number of edges if the graph is triangle-free. Still, such upper bounds can be achieved if one allows larger threshold values $k$. Namely, for an $n$-vertex graph $G$ of maximum degree $\Delta$, we have the following:

- An upper bound $\Theta_k(G) \leq \Delta^2(n/\Delta)^{1/k}$ for every fixed $k$ was proved by Eaton, Gould and Rödl [11].
Remark 1.1. For a bipartite graph \( L \) containing this arc belongs to \( \Theta \) for any graph but, for bipartite graphs, \( \theta \) bound \( nm > L \theta \measures will be denoted by lower case letters \( \theta \). Moreover, we do not require that different vertices receive different sets of labels. The relaxed arcs the intersection rule is satisfied by partition of its vertices. In this case we relax the intersection condition, and only require that can be easily represented just by assigning the same set of labels to all vertices. Note, however, that in these results the requirement that different vertices must receive different sets is essential, since otherwise both \( K \) graph \( \{ \subsets of \} \). For example, the theorem of Frankl and Wilson \[16\] implies that, if \( K \) cannot be strongly represented using \( t \) labels and any type \( L \) of size \( |L| \leq s \). Similar results for bipartite graphs were proved by Frankl and Rödl \[17\], Sgal \[30\] and other authors (see, for example, the references in \[30\]). In particular, Keevash and Sudakov \[22\] have proved that, if \( nm > \sum_{i=0}^{s} \binom{s}{i}2^i \), then a complete bipartite graph \( K_{n,m} \) cannot be strongly represented using \( t \) labels and any type \( L \) of size \( |L| \leq s \). As shown by Sgal \[30\], this bound is tight for \( L = \{0, 1, \ldots, s-1\} \). For single element types \( L = \{k\} \) and sufficiently large values of \( k \), a better bound \( nm > \binom{2s}{k}2^{2s-2k} \), which is about \( 2^t/\sqrt{t} \), was proved by Alon and Lubetzky \[3\]. This proves the conjecture of Ahlswede, Cai and Zhang \[1\], and is also tight for \( t \geq 2k \): assign a unique vertex of \( K_{1,n} \) on the left part a \( 2k \)-element set \( A \), and assign the \( n = \binom{2s}{k}2^{2s-2k} \) vertices on the right part subsets of \( \{1, \ldots, t\} \) that meet \( A \) in \( k \) points. Note, however, that in these results the requirement that different vertices must receive different sets is essential, since otherwise both \( K \) and \( K_{n,m} \) can be easily represented just by assigning the same set of labels to all vertices.

In this paper we are interested in the intersection dimension of bipartite graphs with a fixed partition of its vertices. In this case we relax the intersection condition, and only require that the intersection rule is satisfied by arcs, that is, by pairs of vertices from different parts of the bipartition—the sets of labels of vertices in one part of the bipartition may intersect arbitrarily! Moreover, we do not require that different vertices receive different sets of labels. The relaxed measures will be denoted by lower case letters \( \theta_L(G), \theta_{thr}(G), \theta_{mod}(G) \), etc. Hence, \( \theta_L(G) \leq \Theta_L(G) \) for any graph but, for bipartite graphs, \( \theta_L(G) \) may be much smaller than \( \Theta_L(G) \): for example, \( \Theta_1(K_{n,n}) = n^2 \) but \( \theta_1(K_{n,n}) = 1 \).

Remark 1.1. For a bipartite graph \( G \), \( \theta_L(G) \) is the smallest number \( t \) for which there exist \( t \) complete bipartite graphs \( R_1, \ldots, R_t \) such that an arc is an edge of \( G \) iff the number of the \( R_i \)'s containing this arc belongs to \( L \).
Remark 1.2 (Intersection dimension and boolean functions). The relaxation of intersection rules for bipartite graphs is motivated by their intimate relation to boolean functions: we can look at every bipartite \( n \times n \) graph \( G \) with \( n = 2^m \) as a boolean function \( f_G(z_1, \ldots, z_{2m}) \) in \( 2m \) variables such that \( f_G(\vec{x}, \vec{y}) = 1 \) iff \( x \) and \( y \) are adjacent in \( G \); here \( \vec{x} \in \{0,1\}^m \) is the binary code of \( x \).

Covering of \( G \) by complete bipartite graphs corresponds then to computing the function \( f_G(\vec{x}, \vec{y}) \) by a depth-2 formula of the form

\[
f_G(\vec{x}, \vec{y}) = \text{SYM}(h_1(\vec{x}, \vec{y}), \ldots, h_t(\vec{x}, \vec{y})),
\]

where each \( h_i \) is an AND of some variables and/or their negations, and \( \text{SYM} : \{0,1\}^t \rightarrow \{0,1\} \) is an arbitrary symmetric boolean function, that is, a boolean function whose output only depends on the number of 1’s in the input vector. To find an explicit boolean function in \( m \) variables requiring \( t \geq 2(\log m)^{\alpha} \) ANDs for some \( \alpha \rightarrow \infty \) is an old problem whose solution would have important consequences in computational complexity (see Section 8). Since each \( R_i = \{xy : h_i(\vec{x}, \vec{y}) = 1\} \) forms a bipartite complete graph, any such formula for \( f_G \) must use at least \( t \geq \theta(G) \) ANDs. Hence, what we need are explicit bipartite graphs of large intersection dimension.

Another bridge between bipartite graphs and boolean functions is given by a simple fact that \( \log \theta_1(G) \) is precisely the nondeterministic communication complexity of \( f_G \) (see, for example, [25]).

2 Our results

Easy counting shows (see Section 3.1) that for the maximum \( \theta(n) \) of the absolute dimension \( \theta(G) = \min_L \theta_L(G) \) over all bipartite \( n \times n \) graphs \( G \), we have

\[
(1/2 - o(1))n \leq \theta(n) \leq n - 1.
\]

The same argument yields that \( \theta(G) \geq \Delta \log(n/\Delta) \) for most bipartite \( n \times n \) graphs of maximal degree \( \Delta \).

2.1 A general upper bound

Our first result shows that this is almost optimal. As customary, the average degree of a graph \( G = (V,E) \) is the fraction \( 2|E|/|V| \). Let \( \delta = \delta(G) \) denote the maximum average degree of an induced subgraph of \( G \). Note that \( \delta \) does not exceed the maximum degree, and can be much smaller for some graphs. For a bipartite graph \( G = (U \cup W, E) \), \( \overline{G} = (U \cup W, \overline{E}) \) stands for its bipartite complement with \( \overline{E} = (U \times W) \setminus E \). Note that, for every graph \( G \) and every type \( L \), every representation of \( G \) with respect to \( L \) is also a representation of \( \overline{G} \) with respect to the complementary type \( \overline{L} = \{0,1,\ldots\} \setminus L \).

**Theorem 1.** For every bipartite \( m \times n \) graph \( G \), \( \theta(G) \leq \theta_1(\overline{G}) \leq \delta(G) \log n \).

In this paper we are, however, mainly interested in finding explicit bipartite graphs whose intersection dimension with respect to different types \( L \) is large, at least \( n^\varepsilon \) for a constant \( \varepsilon > 0 \).
Of particular interest would be to prove such a lower bound for the modular dimension \( \theta_{\text{mod}}(G) \) (see Section 8). However, the best we have so far is a lower bound \( \theta_{\text{mod}}(G) \geq \theta(G) \geq \log n \) holding for all twin free graph, where no two vertices have the same set of neighbors: different vertices then require different sets of labels.

### 2.2 Modular dimensions

Recall that \( \theta_{\text{mod}}(G) \) is the minimum of \( \theta_{L}(G) \) over all modular types, that is, over all types of the form \( L = \{ k: k \mod p \in R \} \). The next theorem shows that the main difficulty here is to get rid of large residue classes \( R \), larger than \( \log n \).

**Theorem 2.** Let \( L = \{ k: k \mod p \in R \} \) for some prime number \( p \) and some subset \( R \) of \( r = |R| \) residues. Then for every bipartite \( n \times n \) graph \( G \) of maximum degree \( \Delta \) we have

\[
\theta_{L}(G) \geq p^{-1} \sqrt{\frac{n}{r\Delta}} \quad \text{and} \quad \theta_{L}(G) \geq \sqrt{\frac{n}{r\Delta}}.
\]

As an easy consequence, we have that for any type \( L \) with \( |L| = s > 0 \),

\[
\theta_{L}(G) \geq 2^{s-1} \sqrt{\frac{n}{s\Delta}} \quad \text{and} \quad \theta_{L}(G) \geq \sqrt{\frac{n}{s\Delta}}.
\]

### 2.3 Threshold dimensions

Next, we consider the threshold dimensions of graphs. Eaton and Rödl in [9] observed that \( \theta_{k}(G) \geq \theta_{1}(G)^{1/k} \) for every graph \( G \): having a threshold-\( k \) representation \( \{ A_x: x \in V \} \) of \( G \), we can look at \( k \)-element subsets of labels as new labels, and assign to each vertex \( x \) the set \( A'_x \) of all \( k \)-element subsets of \( A_x \); this gives a threshold-1 representation.

If \( M \) is an \( n \)-matching \( M \), that is, a bipartite \( n \times n \) graph consisting of \( n \) vertex disjoint edges, then clearly \( \theta_{1}(M) = n \), and we obtain \( \theta_{k}(M) \geq n^{1/k} \) for every threshold value \( k \). However, the general upper bound of [9] mentioned above implies that \( \theta_{\text{thr}}(M) \leq \Theta_{\text{thr}}(M) \leq \log n \). This can be also shown directly: if \( \binom{t}{k} \geq n \), then we can assign to both endpoints of each edge of \( M \) its own \( k \)-element subset of \( \{1, \ldots, t\} \).

Hence, at least for some graphs, large threshold values \( k \) may drastically decrease the dimension. So, a natural question is: What graphs have large threshold dimension independent of the used threshold value \( k \)?

We show that such are Hadamard graphs. Recall that an Hadamard matrix of order \( n \) is an \( n \times n \) matrix with entries \( \pm 1 \) and with row vectors mutually orthogonal (over the reals). A graph associated with an Hadamard matrix (or just an Hadamard graph) of order \( n \) is a bipartite \( n \times n \) graph \( H \) where two vertices \( u \) and \( v \) are adjacent if and only if the corresponding entry of the Hadamard matrix is equal \( +1 \).

**Theorem 3.** For every bipartite \( n \times n \) Hadamard graph \( H \), \( \theta_{\text{thr}}(H) \geq \sqrt{n} \).
The bounds above imply that the parity and threshold dimensions are incomparable, and the
gaps may be even exponential in both directions. For this it is enough to compare the corresponding
intersection dimensions of an $n \times n$ matching $M$ and of a bipartite $n \times n$ Sylvester graph $G$; recall
that $S$ is a Hadamard graph with $n = 2^r$ vertices of each color identified with subsets of $\{1, \ldots, r\}$,
and two vertices $x$ and $y$ are adjacent iff $|x \cap y|$ is odd. Hence, $\theta_{odd}(S) \leq r = \log n$. Moreover,
$\theta_{odd}(M) = n$ since the parity dimension of every bipartite graph is just the rank of its adjacency
matrix over $GF(2)$.

**Corollary 1.** We have $\frac{\theta_{odd}(M)}{\theta_{thr}(M)} \geq n/\log n$ and $\frac{\theta_{thr}(S)}{\theta_{odd}(S)} \geq \sqrt{n}/\log n$.

The example of Sylvester graphs shows another interesting fact: some Ramsey graphs have very
small parity dimension. Namely, Pudlák and Rödl show in [28] that $S$ contains an induced $\sqrt{n} \times \sqrt{n}$
subgraph which is Ramsey, meaning that neither the graph nor its complement contains a copy of $K_{s,s}$, for $s$ much larger than $\log n$. Since the intersection dimension of induced subgraphs does not exceed that of the original graph, this implies that some bipartite Ramsey graphs have logarithmic
parity dimension.

Threshold dimensions of graphs were also considered in the context of the randomized com-
munication complexity. Namely, given a predicate $D : \{0, 1, \ldots, r\} \rightarrow \{0, 1\}$, one may define a
Sylvester-type graph $S_D$ as a bipartite $n \times n$ graph with $n = 2^r$ vertices of each color identified
with subsets of $\{1, \ldots, r\}$, and two vertices $x$ and $y$ are adjacent iff $D(|x \cap y|) = 1$. The *alternation
number* $alt(D)$ of $D$ is the number of $i$ for which $D(i) \neq D(i - 1)$. Hence, Sylvester graphs $S$
correspond to the case when $D(i) = i \mod 2$ is the parity predicate with $alt(D) = r/2$. Using involved
algebraic arguments, Forster [15] and then Sherstov [31] were able to derive a more general lower
bound $\log \theta_{thr}(S_D) \geq \frac{alt(D)}{\log \log n}$ holding for any predicate $D$. Still, our proof of Theorem 3
has an advantage that it only uses relatively simple averaging arguments, and hence, has a potential
to be extended to other types $L$, like the interval types of the form $L = \{a, a + 1, \ldots, b\}$. This
could be a first step to handle the modular intersection dimension $\theta_{mod}(G)$ (see Section 8 for a
discussion).

**2.4 Balanced representations**

When trying to prove large lower bounds (larger than $\log n$) on the absolute intersection dimension
$\theta(G) = \min_L \theta_L(G)$ of explicit graphs, we are faced with two problems: the adversary is allowed

(i) to choose an arbitrary type $L$, and

(ii) to assign vertices $x$ arbitrary sets $A_x$.

If both are allowed, we cannot prove nothing more than $\theta(G) \geq \log n$. The lower bounds above
correspond to the case when we allow (ii) but restrict (i). Now we look at what happens if we allow
(i) but restrict (ii).

The most “natural” intersection representation of any (non necessarily bipartite) graph is to
assign each vertex $x$ the set $A_x$ of its incident edges. This gives a threshold-1 representation of the
graph. The representation itself has, however, an additional property that \(|A_u \cap A_v \cap A_x| = 0\) for any triple of distinct vertices.

Motivated by this example, we say that a representation of a bipartite graph is **balanced** if there are two vertices \(x \neq y\) on the left part such that \(|A_u \cap A_v \cap A_x| = |A_u \cap A_v \cap A_y|\) for all pairs of vertices \(u \neq v\) on the right part.

It is easy to see that every bipartite \(n \times n\) graph has a balanced threshold-1 representation using at most \(n\) labels: assign to each vertex \(x\) on the left part the set \(A_x\) of its neighbors, and assign to each vertex \(u\) on the right part a single element set \(A_u = \{u\}\). This is an intersection representation with respect to the type \(L = \{1\}\). A natural question is: Can the number of labels be substantially reduced by using another types \(L\)? Our next result says that, at least for Hadamard graphs, this is not possible.

**Theorem 4.** Every balanced representation of a bipartite \(n \times n\) Hadamard graph must use at least \(n/4\) labels.

### 2.5 The weight of representations

So far we were interested in the size \(|\bigcup_{x \in V} A_x|\) of representations \(\{A_x: x \in V\}\), that is, in the total number of used labels. Another important measure of representations is their **weight** \(\sum_{x \in V} |A_x|\). Let \(w_L(G)\) denote the weight analog of \(\theta_L(G)\), that is, the smallest weight of an intersection representation of \(G\).

These measures were mainly considered with respect to the threshold-1 type \(L = \{1, 2, \ldots\}\), and to the parity type consisting of all odd natural numbers. The interest in this last type is motivated by the fact that \(w_{\text{odd}}(G)\) is precisely the smallest number of wires in a depth-2 circuit with unbounded fanin parity gates computing the linear transformation \(Ax = y\) over \(GF(2)\), where \(A\) is the adjacency matrix of \(G\) (see [4]). The following was known:

- \(w_1(G) \leq n^2 / \log n\) for every \(n\)-vertex graph \(G = (V, E)\), and
- \(w_1(G) \geq |E|/r\) if \(G\) is \(K_{r,r}\)-free (Chung, Erdős and Spencer [7]).
- \(w_{\text{odd}}(H) \geq n \log n\) for any bipartite \(n \times n\) Hadamard graph \(H\) (Alon, Karchmer and Wigderson [4]).

For Hadamard graphs, the last lower bound is almost optimal, because \(w_{\text{odd}}(S) \leq n \log n\) for a bipartite \(n \times n\) Sylvester graph \(S\). We show that the argument of [4] can be extended to arbitrary types.

**Theorem 5.** For every bipartite \(n \times n\) Hadamard graph \(H\) and for every type \(L \subseteq \{0, 1, \ldots\}\), we have that \(w_L(H) \geq n \log n / \log \log n\).
2.6 Intersection representations and the Log-Rank Conjecture

Finally, we prove one result concerning the so-called “Log-Rank Conjecture” in communication complexity (see, for example, [25]). The conjecture states that the deterministic communication complexity of any 0/1 matrix is at most poly-logarithmic in its real rank. Nisan and Wigderson [25] proved that this conjecture is equivalent to the following conjecture about bipartite graphs, where \( \text{rk}(G) \) denotes the rank of the 0/1 adjacency matrix of \( G \) over the field of real numbers, and \( \omega(G) \) is the maximum number of edges in a complete bipartite graph lying in \( G \) or in \( \overline{G} \).

**Conjecture 1** (Nisan–Wigderson [25]). There exists a constant \( c > 0 \) such that, for every bipartite \( n \times m \) graph \( G \), \( \omega(G) \geq \frac{nm}{2^{\log \text{rk}(G)}} \).

They have also supported the conjecture by proving that every graph of small rank must have large discrepancy. The discrepancy \( \text{Disc}(G) \) of a graph \( G \) is the maximum, over all complete bipartite graphs \( R \), of the absolute value of the difference between the number of edges and the number of non-edges of \( G \) lying in \( R \). Note that \( \text{Disc}(G) \geq \omega(G) \).

**Theorem 6** (Nisan–Wigderson [25]). For every bipartite \( n \times m \) graph \( G \), we have \( \text{Disc}(G) \geq \frac{nm}{\text{rk}(G)^{3/2}} \).

Moreover, the bond in this theorem is nearly tight: for every \( r \) there are infinitely many \( n \times n \) graphs \( G \) with \( \text{rk}(G) = r \) and \( \text{Disc}(G) \leq \frac{n^2}{r} \). This can be shown by taking square arrays of \( r \times r \) Hadamard matrices.

Still, Conjecture 1, and hence, the Log-Rank Conjecture remain widely open. The real rank of a 0/1 matrix \( A \) is the smallest number of real rank 1 matrices summing up to \( A \). To approach Conjecture 1, Sgal [30] suggested to first solve it for special kind of rank \( r \) 0/1 matrices \( A \) representable as a sum of \( r \) rank 1 0/1 matrices. Since these last matrices are adjacency matrices of complete bipartite graphs, we arrive to the following question concerning intersection representations of graphs.

A \( k \)-tight representation of a graph is an intersection representation of type \( L = \{ k \} \) with an additional condition that \( |A_x \cap A_y| \in \{ k-1, k \} \) for all arcs \( xy \) of \( G \). The intersection matrix of each such representation has entries \( k-1 \) and \( k \) only, and if we subtract \( k-1 \) from each entry, we obtain a 0/1 matrix whose rank is at most 1 plus the number of used labels. A representation is **tight** if it is \( k \)-tight for some \( k \geq 1 \).

**Conjecture 2** (Sgal [30]). There exists a constant \( c > 0 \) such that, if a bipartite \( n \times m \) graph \( G \) has a tight representation using \( r \) labels, then \( \omega(G) \geq \frac{nm}{2^{\log r^c}} \).

We show that Sgal’s conjecture is true for all \( k \)-tight representations, as long as \( k \) is at most poly-logarithmic in the total number of used labels.

**Theorem 7.** If a bipartite \( n \times m \) graph \( G \) has a \( k \)-tight representation using \( r \) labels, then \( \omega(G) \geq \frac{nm}{4^{k \log r + 1}} \).

Now we turn to the proofs.
3 Proofs

3.1 Proof of Theorem 1

We have at most $2^{2t^n}$ possible encodings of $2n$ vertices by subsets of $\{1, \ldots, t\}$, and at most $2^{t+1}$ possibilities to choose the type $L \subseteq \{0, 1, \ldots, t\}$. Hence, at most $2^{2t^n+t+1}$ bipartite $n \times n$ graphs can have intersection dimension at most $t$. On the other hand, we have $2^n$ such graphs, implying that some of them need $t \geq (1/2 - o(1))n$ labels. Since we have at least $(n/\Delta)^n\Delta/2$ bipartite $n \times n$ graphs of maximum degree at most $\Delta$ (see, for example, Proposition 2.1 in [9]), this also implies that some of degree-$\Delta$ graphs require $t \geq \Delta \log (n/\Delta)$ labels, independent on what type $L$ we use.

This proves the statements we made before Theorem 1. Now we turn to the proof of the theorem itself. Given a bipartite graph with a fixed bipartition of its vertices into the left and right part, its left (resp., right) degree is the maximum degree of a vertex in the corresponding part of the bipartition.

**Lemma 1.** Let $G$ be a bipartite $m \times n$ graph such that each vertex on the right part has degree at most $d$. Then $\theta_1(G) \leq ed \ln mn$.

**Proof.** We can construct a complete bipartite subgraph $S \times T$ of $\overline{G}$ via the following probabilistic procedure: pick every vertex $x$ on the left part independently, with probability $p = 1/d$ to get a random subset $S$, and let $T$ be the set of all those vertices $y$ on the right part that are adjacent to all vertices in $S$. An edge $xy$ of $\overline{G}$ is covered by such a subgraph $S \times T$ if $x$ was chosen in $S$ and none of (at most $d$) neighbors of $y$ in $G$ was chosen in $S$. Hence, this happens with probability at least $p(1-p)^d \geq pe^{-pd} = p/e$. If we apply this procedure $t$ times to get $t$ complete bipartite subgraphs, then the probability that $xy$ is covered by none of these subgraphs does not exceed $(1-p/e)^t \leq e^{-tp/e}$. Hence, the probability that some edge of $\overline{G}$ remains uncovered is smaller than $mne^{-tp/e} = \exp(\ln mn - t/(ed))$, which is < 1 for $t = ed \ln mn$. \hfill \Box

We will also use the following lemma proved in [29]. In an oriented graph, the notation $d_+(x)$ stands for the out-degree of $x$.

**Lemma 2** (Reiterman–Rödl–Sinajová[29]). Every graph $G = (V,E)$ admits an orientation of the edges with $d_+(x) \leq \delta(G)$ for all $x \in V$.

**Proof.** Induction on $n = |V|$. For $n = 1$ the result is trivial. When $n > 1$, we select a vertex $x \in V$ of degree at most $\delta(G)$. This can be done since the average degree of the graph $G$ itself must be at most $\delta(G)$. Orient now the edges of $G - x$ using the induction assumption, and replace each edge $xy \in E$ by an oriented edge going from $x$ to $y$. The out-degree of vertices in $G - x$ remains unchanged, and that of $x$ does not exceed $\delta(G)$. \hfill \Box

By Lemma 2, every bipartite graph $G$ can be written as union of two bipartite graphs such that the left degree of the first graph as well the right degree of the second graph does not exceed $\delta(G)$: the first graph contains all edges of $G$ oriented from left to right, and the second consists of all remaining edges. Hence, Theorem 1 follows from Lemma 1.
3.2 Proof of Theorem 2

When trying to estimate the intersection dimension of a bipartite $n \times n$ graph $G$, we are faced with the following problem. We have two systems $\mathcal{A} = \{A_1, \ldots, A_n\}$ and $\mathcal{B} = \{B_1, \ldots, B_n\}$ of (not necessarily distinct) subsets of $\{1, \ldots, t\}$ and (if the type $L$ of the intersections is not given) the only knowledge about these systems is that the intersection sizes $|A_i \cap B_j|$ must be consistent with a given graph $G$: the pairs $A_i, B_j$ corresponding to edges and to non-edges must have different intersection sizes. Hence, the whole information about the pair $\mathcal{A}, \mathcal{B}$ we are interested in is given by its intersection matrix $I(\mathcal{A}, \mathcal{B}) = \{|A_i \cap B_j| : 1 \leq i, j \leq n\}$. Since $I(\mathcal{A}, \mathcal{B})$ is a matrix of scalar products of the corresponding characteristic vectors, the size $t$ of the universum is at least the rank of $I(\mathcal{A}, \mathcal{B})$ over the reals. Hence, the most direct (and most difficult) way would be to try to prove that the intersection matrix $I(\mathcal{A}, \mathcal{B})$ of every pair $\mathcal{A}, \mathcal{B}$ of set-systems representing $G$ must have large rank.

Another, less direct approach could be to try to use the properties of a given graph $G$ to construct a function $f : \{0,1\}^t \to \mathbb{F}$ for some field $\mathbb{F}$, and to show that the $f$-intersection matrix

$$I_f(\mathcal{A}, \mathcal{B}) = \{f(A_i \cap B_j) : 1 \leq i, j \leq n\}$$

must have large rank over $\mathbb{F}$; here and in what follows, $f(C)$ denotes the value of $f$ on the incidence $0/1$ vector of $C \subseteq \{1, \ldots, t\}$. To conclude that then the number $t$ of used labels must be large, we only need that $f$ has small “weight”.

By a weight $w(f)$ of a function $f : \mathbb{F}^t \to \mathbb{F}$ we mean the smallest number $w$ such that, when restricted to the binary cube $\{0,1\}^t$, the function $f$ can be written as a linear combination $f(x_1, \ldots, x_t) = a_1X_1 + \cdots + a_wX_w$ over $\mathbb{F}$ of $w$ monomials, that is, products $X_i = \prod_{j \in I} x_j$ of variables; for $I = \emptyset$ we assume that the product is equal 1. In particular, every multi-linear polynomial over $\mathbb{F}$ with $w$ nonzero coefficients has weight at most $w$. Note that the polynomial $f = x_1 + x_2 + \cdots + x_t$ has weight $t$, and in this case we have that $I_f(\mathcal{A}, \mathcal{B}) = I(\mathcal{A}, \mathcal{B})$. Finally, every function $f : \mathbb{F}^t \to \mathbb{F}$ has weight at most $2^t$ since, for every two vectors $a = (a_1, \ldots, a_t)$ and $x = (x_1, \ldots, x_t)$ in $\{0,1\}^t$, the value of $\prod_{i=1}^t x_i^{a_i}$ with $x^1 = x$ and $x^0 = 1 - x$ equals 1 for $x = a$, and equals 0 for $x \neq 0$.

**Lemma 3.** For every function $f : \mathbb{F}^t \to \mathbb{F}$, every $f$-intersection matrix has rank at most $w(f)$ over $\mathbb{F}$.

**Proof.** Let $\mathcal{A}$ and $\mathcal{B}$ be systems of subsets of $\{1, \ldots, t\}$, and let $f : \mathbb{F}^t \to \mathbb{F}$ be a function of weight $w(f) = w$. Then the restriction of $f$ to the binary cube $\{0,1\}^t$ can be written as a linear combination $f = a_1X_1 + \cdots + a_wX_w$. Since for every product $X_I = \prod_{i \in I} x_i$, we have

$$X_I(A \cap B) = 1 \text{ iff } I \subseteq A \cap B \text{ iff } I \subseteq A \text{ and } I \subseteq B \text{ iff } X_I(A) \cdot X_I(B) = 1,$$

the value $f(A \cap B)$ is just the scalar product of two vectors

$$(a_1X_1(A), \ldots, a_wX_w(A)) \text{ and } (X_1(B), \ldots, X_w(B))$$

of length $w$ over $\mathbb{F}$, implying that the rank of $I_f(\mathcal{A}, \mathcal{B})$ cannot exceed $w = w(f)$. \end{proof}
A bipartite $n \times n$ graph is increasing if its adjacency matrix $A = (a_{ij})$ satisfies $a_{ii} = 1$ for all $i$, and $a_{ij} = 0$ for all $i > j$. In particular, every matching is an increasing graph.

If $G$ is a bipartite $n \times n$ graph of maximum degree $\Delta$ (and with no isolated vertices), then one can obtain an induced increasing $(n/\Delta) \times (n/\Delta)$ subgraph of $G$ by inductively removing edges and all the neighbors of their right endpoints. Hence, Theorem 2 is a direct consequence of the following

**Theorem 8.** Let $G$ be an increasing bipartite $n \times n$ graph, $p$ a prime number and $1 \leq r < p$ an integer. Then for every type of the form $L = \{k: k \mod p \in R\}$ with $|R| \leq r$ we have $\theta_L(G) \geq (n/r)^{1/(p-1)} - 1$ and $\theta_L(G) \geq n^{1/r} - 1$.

**Proof.** Let $G$ be an arbitrary increasing bipartite $n \times n$ graph, and let $A = \{A_1, \ldots, A_n\}$ and $B = \{B_1, \ldots, B_n\}$ be systems of subsets of $\{1, \ldots, t\}$ associated with its vertices. Suppose that these two systems form an intersection representation of $G$ with respect to a type $L = \{k: k \mod p \in R\}$ with $|R| \leq r$. When taken modulo $p$, the diagonal entries of the intersection matrix $I(A, B)$ must then belong to $R$, and none of the entries below the diagonal can belong to $R$. We can therefore find a number $a \in R$ and a subset $I \subseteq \{1, \ldots, n\}$ of size $|I| \geq n/|R| = n/r$ such that $|A_i \cap B_i| \mod p = a$ for all $i \in I$, and $|A_i \cap B_j| \mod p \neq a$ for $i > j$. If we take a polynomial

$$g_a(z_1, \ldots, z_y) = z_1 + \cdots + z_t - a$$

then, modulo $p$, the corresponding $g_a$-intersection submatrix has zeroes on the diagonal, and has nonzero entries below the diagonal. Hence, if we take

$$f(\vec{z}) = 1 - g_a(\vec{z})^{p-1}$$

then, by Fermat’s little theorem, the corresponding $f$-intersection matrix is a lower triangular matrix with nonzero diagonal entries, and must therefore have full rank $|I| \geq n/r$ over $GF(p)$. Lemma 3 implies $n/r \leq |I| \leq w(f) \leq (t + 1)^{p-1}$, and the lower bound $t \geq (n/r)^{1/(p-1)} - 1$ follows.

If $A, B$ is an intersection representation of the complement graph $\overline{G}$ with respect to the same type, then this is an intersection representation of the graph $G$ itself with respect to the type $\{k: k \mod p \notin R\}$. Hence, if we take

$$h(\vec{z}) = \prod_{a \in R} g_a(\vec{z})$$

then, modulo $p$, the $h$-intersection matrix $I_h(A, B)$ itself is a lower triangular matrix with nonzero diagonal entries, and must therefore have full rank $n$ over $GF(p)$. Lemma 3 implies in this case $n \leq w(h) \leq (t + 1)^r$, and the desired lower bound $t \geq n^{1/r} - 1$ follows.

Since for every natural number $s$ there is a prime $p$ with $s < p \leq 2s$, Theorem 8 implies

**Corollary 2.** If $|L| = s$, then every intersection representation of an increasing bipartite $n \times n$ graph with respect to $L$ requires at last $(n/s)^{1/(2s-1)} - 1$ labels, and that of a complement graph requires at least $n^{1/s} - 1$ labels.
4 Proof of Theorem 3

A standard tool when dealing with threshold-type measures is to introduce a particular notion of "discrepancy" (see, for example, [19]). Following this tradition, we define the relative discrepancy, \(\text{disc}(G)\), of a graph \(G\) as the smallest number \(\delta\) such that the proportion of edges and of non-edges of \(G\) in an arbitrary complete graph differ by at most \(\delta\).

**Lemma 4.** For every bipartite graph \(G\) we have \(\theta_{\text{thr}}(G) \geq 1/\text{disc}(G)\).

**Proof.** Let \(G = (V, E)\) be a bipartite graph, and let \(R_1, \ldots, R_t \subseteq V\) be a threshold-\(k\) covering of \(G\) by complete bipartite graphs. That is, an arc is an edge in \(G\) iff it belongs to at least \(k\) of the graphs \(R_i\)'s. Since every element of \(E\) belongs to at least \(k\) of the sets \(E \cap R_i\), the average size of these sets must be at least \(k\). Since no element of \(E\) belongs to more than \(k-1\) of the sets \(E \cap R_i\), the average size of these sets must be at most \(k-1\). Hence, the difference of these average sizes is at least 1. Since, on the other hand, we have only \(t\) graphs \(R_i\), this difference cannot be larger than \(t \cdot \text{disc}(G)\), and the lower bound \(t \geq 1/\text{disc}(G)\) follows.

Hence, Theorem 3 follows from Lemma 4 and from

**Lemma 5.** If \(H\) is a bipartite \(n \times n\) graph, then \(\text{disc}(H) \leq n^{-1/2}\).

**Proof.** The number of edges as well as of non-edges of \(H\) is at least \(cn^2\) for a constant \(c > 0\). The well-known Lindsey's lemma (see, for example, [13], p. 88) says that the sum of all entries in any \(s \times t\) submatrix of an \(n \times n\) Hadamard matrix lies between \(-\sqrt{stn}\) and \(\sqrt{stn}\). In particular, this sum lies between \(-n^{3/2}\) and \(n^{3/2}\) for any submatrix. In terms of graphs, this means that the difference between the number of edges and the number of non-edges of \(H\) in any complete bipartite graph must lie between \(-n^{3/2}\) and \(n^{3/2}\). Since the numbers of edges as well as of non-edges of \(H\) lie between \(cn^2\) and \(n^2\), this means that \(\text{disc}(H) \leq n^{3/2}/n^2 = n^{-1/2}\).

5 Proof of Theorem 4

We will lower-bound the number of labels in balanced representations in terms of the following characteristic of graphs, which we will also use in the next section.

A bipartite graph is \(k\)-isolated if, for every two vertices \(x \neq y\) on the left part, there is a set \(S\) of \(|S| = k\) vertices on the right part such that every vertex \(u \in S\) is adjacent to \(x\) and non-adjacent to \(y\). For example, a bipartite Hadamard \(n \times n\) graph is \(k\)-isolated with \(k \geq n/4\). Hence, Theorem 4 is a special case of the following

**Theorem 9.** Every balanced representation of a \(k\)-isolated bipartite \(n \times n\) graph must use at least \(k\) labels.

**Proof.** Let \(A = \{A_x : x \in V\}\) be a balanced intersection representation of \(G\) using \(t\) labels. Let \(V = V_1 \cup V_2\) be the bipartition of \(G\). Since the representation is balanced, there must exist two vertices \(x \neq y \in V_1\) such that their sets of labels \(X = A_x\) and \(Y = A_y\) satisfy

\[
|A_u \cap A_v \cap X| = |A_u \cap A_v \cap Y|
\]

for all \(u \neq v \in V_2\).
On the other hand, since the graph is \( k \)-isolated, there must be a subset \( S \subseteq V_2 \) of \( |S| = k \) vertices such that every vertex \( u \in S \) is adjacent to \( x \) and non-adjacent to \( y \). Hence,

\[
|A_u \cap X| \neq |A_u \cap Y| \quad \text{for all } u \in S.
\]

For every subset \( C \subseteq \{1, \ldots, t\} \), the value \( f(C) \) of a real polynomial

\[
f(z_1, \ldots, z_t) = \sum_{i \in X} z_i - \sum_{i \in Y} z_i
\]

is the difference between \( |C \cap X| \) and \( |C \cap Y| \). Hence, by what was said, \( f(A_u \cap A_v) = 0 \) for all \( u \neq v \in S \), and \( f(A_u \cap A_v) \neq 0 \) for all \( u \in S \). That is, the \( f \)-intersection matrix \( I_f(A') \) of \( A' = \{A_u : u \in S\} \) is a real diagonal matrix with nonzero diagonal entries. Lemma 3 implies \( t \geq w(f) \geq |S| = k \).

6 Proof of Theorem 5

Theorem 5 is a direct consequence of the following

**Theorem 10.** If a bipartite \( n \times n \) graph \( G \) is \( k \)-isolated, then \( w_L(G) \geq k \log n / \log \log n \) for every type \( L \)

**Proof.** Let \( A_1, \ldots, A_n, B_1, \ldots, B_n \) be an intersection representation of \( H \) with respect to some type \( L \). Let \( m = c \log n / \log \log n \) for a sufficiently small absolute constant \( c > 0 \). If \( \sum_{i=1}^{n} |A_i| > mn \), then we are done. So, assume that \( \sum_{i=1}^{n} |A_i| \leq mn \). Our goal is to show that then \( \sum_{j=1}^{n} |B_j| \geq mk \).

A well-known result of Erdős and Rado [9] says that every family of \( r! s^r \) sets, each of which has cardinality less than \( r \), contains a sunflower with \( s \) petals, that is, a family \( F_1, \ldots, F_s \) of sets of the form \( F_i = P_i \cup C \), where the \( P_i \)'s are pairwise disjoint; the set \( C \) is the core of the sunflower, and the \( P_i \)'s are called the petals.

Now, if \( \sum_{i=1}^{n} |A_i| \leq mn \), then at least \( n/2 \) of the sets \( A_i \) must be of size at most \( r = 2m \). By the sunflower theorem, these sets must contain a sunflower with \( s = 2m \) petals. Having such a sunflower with a core \( C \), we can pair its members arbitrarily, \((A_{u_1}, A_{v_1}), \ldots, (A_{u_m}, A_{v_m})\); hence, \( A_{u_i} \cap A_{v_i} = C \) for all \( i = 1, \ldots, m \).

Since the graph is \( k \)-isolated, each two vertices \( u_i \neq v_i \) have a set \( S \) of \( |S| = k \) vertices on the other side, all of which are adjacent to \( u_i \) and none of which is adjacent to \( v_i \). Hence, independent on the type \( L \), we have that \( |A_{u_i} \cap B_j| \neq |A_{v_i} \cap B_j| \) for all \( j \in S \). Since clearly \( |A_{u_i} \cap C| = |A_{v_i} \cap C| \), this implies that \( B_j \) must have at least one element in the symmetric difference \( D_i = A_{u_i} \oplus A_{v_i} \). Hence, \( \sum_{j=1}^{n} |B_j \cap D_i| \geq |S| = k \) for each \( i = 1, \ldots, m \). Since the sets \( D_1, \ldots, D_m \) are disjoint, this implies \( \sum_{j=1}^{n} |B_j| \geq mk \).

7 Proof of Theorem 7

Since every \( k \)-tight representation is also a threshold-\( k \) representation, Theorem 7 is a direct consequence of the following
Theorem 11. Let $G$ be a bipartite $n \times m$ graph of threshold-$k$ dimension $r$. Then either $G$ contains a bipartite clique with $nm/4$ edges or $G$ contains a bipartite clique with $nm/4\left(\binom{r}{k}\right)^2$ edges.

Proof. Let $\{A_x : x \in V\}$ be a threshold-$k$ representation of $G$ using $r$ labels. We say that a set of labels $S$ appears in a vertex $x$ if $S \subseteq A_x$. We also say that a set $S$ is left popular (resp., right popular) if it appears in at least $\alpha = 1/2\left(\binom{r}{k}\right)$ fraction of vertices on the left (resp., on the right) part of the bipartition.

If at least one $k$-element set $S$ of labels is left popular as well as right popular, then this set appears in at least $\alpha n$ vertices on the left part as well as in at least $\alpha m$ vertices on the right part. The corresponding vertices all share the set $S$, and hence, form a bipartite clique in $G$ with at least $\alpha^2 nm$ edges.

Assume now that no $k$-element set $S$ of labels is both left and right popular. If a set $S$ is not left popular, then it can appear in at most $\alpha n$ of vertices on the left part. Hence, the number of vertices on the left part containing at least one left popular $k$-element subset of labels does not exceed $\binom{r}{k} \cdot \alpha n = n/2$. We can therefore find a subset $L$ of $|L| \geq n/2$ vertices $x$ on the left part, all whose $k$-element subsets $S \subseteq A_x$ are left popular. By symmetry, we can find a subset $B$ of $|B| \geq m/2$ vertices on the right part, all whose $k$-element subsets $S \subseteq A_x$ are right popular. Since, by our assumption, no $k$-element subset of labels can be both left and right popular, no such subset can be contained in any intersection $A_x \cap A_y$ with $x \in L$ and $y \in R$. Hence, $|A_x \cap A_y| < k$ for all $x \in L$ and $y \in R$, that is, in this case the complement graph $\overline{G}$ contains a bipartite $(n/2) \times (m/2)$ clique.

8 Conclusion and open problems

As mentioned in the introduction (Remark 1.2), high lower bounds on the intersection dimension of explicit bipartite graphs would resolve some old problems in the computational complexity of boolean functions. In this paper we obtained such lower bounds when either the form of the type $L$ or the form of used sets of labels is restricted. Our results, as well as previous ones, are still too weak to have new consequences for boolean functions. Below we shortly describe what we need to have such consequences. In all these problems we are looking for an explicit sequence of bipartite $n \times n$ graphs of large intersection dimension.

Problem 1. Prove $\theta_{\text{mod}}(G) \geq 2^{\left(\log \log n\right)^\alpha}$ for some $\alpha \to \infty$.

By results of result of Yao [33] and Beigel and Tarui [6], this would yield the first superpolynomial lower bound for constant depth circuits with arbitrary modular gates. Actually, it would be enough to prove such a lower bound on $\theta_L(G)$ for a special kind of modular types $L$ consisting of all natural numbers whose binary representations have a 1 in the middle. Such types (called middle-bit types) consist of disjoint intervals of consecutive numbers.

To approach this question, it would be interesting to first prove large lower bounds for types of the form $L_{a,b} = \{a, a+1, \ldots, b\}$ with $b-a > \log n$. Each such type is an intersection $L_{a,b} = L_{\geq a} \cap L_{\leq b}$ of two threshold types $L_{\geq a} = \{a, a + 1, \ldots\}$ and $L_{\leq b} = \{0, 1, \ldots, b\}$. We already know (see
Theorem 3) that, with respect to both these types, Hadamard $n \times n$ graphs $H$ must have dimension about $\sqrt{n}$. Can the dimension of $H$ with respect to $L_{a,b}$ be much smaller than $\sqrt{n}$?

**Problem 2.** What is the intersection dimension of Hadamard graphs with respect to interval types?

In the context of boolean functions, the next important measure is the following generalization of the threshold-1 dimension $\theta_1(G)$ of graphs. Namely, let $r_s(G)$ be the the smallest number $r$ such that $G$ can be written as an intersection of at most $s$ graphs $G_1, \ldots, G_s$ with $\theta_1(G_i) \leq r$ for all $i = 1, \ldots, s$. That is, in order to reduce the number of complete subgraphs in the covering, we now allow to replace up to a $1 - 1/s$ fraction of non-edges of $G$ by new edges. Let $r(G) = \min_s r_s(G)$.

**Problem 3.** Prove $r(G) \geq n^\varepsilon$ for a constant $\varepsilon > 0$.

Together with the well-known reduction of log-depth circuits to depth-3 circuits, due to Valiant [32], this would give the first super-linear lower bound for log-depth circuits, thus resolving a more than 30 years old open problem in complexity theory (see [20] for more details).

Easy counting shows that $r(G) \geq \sqrt{n}$ holds for almost all bipartite $n \times n$ graphs. The problem, however, is to prove a comparable lower bound for an explicit sequence of graphs. The best we can do so far is a lower bound $r(H) \geq \log^{3/2} n$ proved by Lokam [23] for Hadamard graphs. It was also shown in [20] that $r_s(H) \geq \max\{\sqrt{n}/2^s, n^{1/2s}\}$ for every $s$. Hence, it is important to understand what happens, when $s$ approaches the border of $\log n$. That this may be indeed a critical border can be seen on an example of an $n$-matching $M$: then $r_1(M) = n$ but $r_s(M) \leq 2$ for $s = \log n$. To see this, encode each vertex $x$ on the left part by its own vector $\bar{x} \in \{0,1\}^s$, and assign each matched vertex on the left part the same vector. We can then write $M$ as an intersection of $s$ graphs $G_1, \ldots, G_s$, where $G_i$, consists of all edges whose endpoints have the same bit in the $i$-th coordinate. Since each $G_i$ is a union of just two complete bipartite graphs, $r_s(M) \leq 2$ follows.

**Problem 4.** What is $r_s(H)$ for an $n \times n$ Hadamard graph $H$, when $s = \log n$?

Even a mere existence of a graph $G$ such that $r(G)$ is much larger than $r(G)$ is not known.

**Problem 5.** Does there exists $G_n$ with $\log \log r(G_n)/\log \log r(G_n) \to \infty$.

This would separate the second level of the communication complexity hierarchy and resolve an old problem raised by Babai, Frankl and Simon in [5].

The last problem concerns the weight $w_{\mathrm{odd}}(G)$ of intersection representations with respect to the type consisting of all odd integers.

**Problem 6.** Prove $w_{\mathrm{odd}}(G) \geq n^{1+\varepsilon}$ for a constant $\varepsilon > 0$.

The highest known lower bound $w_{\mathrm{odd}}(G) \geq n \log^{3/2} n$ is due to Pudlák and Rödl [27]. When dealing with this measure, the following equivalent reformulation could be useful: $w_{\mathrm{odd}}(G)$ is the smallest number $w$ for which the adjacency matrix of $G$ can be written as a product $AB$ of two 0-1 matrices $A$ and $B$ over $GF(2)$ such that the total number of nonzero entries in $A$ and $B$ does not exceed $w$. An indication that Hadamard graphs may be not good for this purpose is given in [27]: Sylvester matrix can be decomposed into the product of three matrices with only linear number of nonzero elements.
References