A NOTE ON INDUCED CYCLES IN KNESER GRAPHS

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Let \( g(n, r) \) be the maximal order of an induced cycle in the Kneser graph \( Kn([n], r) \), whose vertices are the \( r \)-sets of \( [n] = \{1, \ldots, n\} \) and whose adjacency relation is disjointness. Thus \( g(n, r) \) is the largest \( m \) for which there is a sequence \( A_1, A_2, \ldots, A_m \subseteq [n] \) of \( r \)-sets with \( A_i \cap A_j = \emptyset \) if and only if \( |i - j| = 1 \) or \( m = 1 \). We prove that there is an absolute constant \( c > 0 \) for which

\[
g(2r + 1, r) > c(2.587)^r,
\]

improving previous results. Our lower bound also shows that the clique covering number of the complement of an \( n \)-cycle is at most \( 1.459 \log_2 n \) for large enough \( n \). Related problems concerning the order of induced subgraphs of bounded degree of Kneser graphs are discussed.

1. Introduction

Given an integer \( r \geq 1 \) and a set \( X \), we define the \( r \)-Kneser Graph on the ground set \( X \), denoted \( Kn(X, r) \), to be the graph whose vertices are the \( r \)-subsets of \( X \), and whose adjacency relation is disjointness. For instance, if \( |X| = 5 \) and \( r = 2 \), then \( Kn(X, r) \) is the complement of the line graph of the complete graph on 5 vertices, i.e. the Petersen graph. These graphs are very natural objects and they have attracted much attention owing to a conjecture of Kneser [14], who gave a plausible value for their chromatic number. Lovász [15], using some algebraic topology, settled the then 23-year-old problem of Kneser in 1978, and Bárány [4] gave an elegant shorter proof soon afterwards. Some other references concerning Kneser graphs are [6], [10] and [16].

In this note we shall study the induced paths and cycles of Kneser graphs. For integers \( n \) and \( r \geq 1 \), let us denote by \( g(n, r) \) the maximum order of an induced cycle of \( Kn([n], r) \), where as usual \( [n] = \{1, \ldots, n\} \). Also, let us set \( g(r) \) to be the corresponding supremum for \( Kn(N, r) \). Clearly \( g(n, r) \) is non-decreasing in \( n \) and \( g(r) = \lim_n g(n, r) \). It turns out that this limit is finite for all \( r \), and in fact the following bounds hold.

\[
2 + 2^r \leq g(r) = \max_n g(n, r) \leq 1 + \left(\frac{2^r}{r}\right).
\]

The upper bound has been observed by several authors: Alles and Poljak [1], Alon [2], de Caen, Gregory and Pullman [7] and Tuza (see [1]); in fact, it follows from a result of Frankl [9] and Kalai [13]. The lower bound was proved by Alles by an inductive argument (see [11]). Also, Theorem 3.1 in [7] essentially gives the

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necessary techniques to prove a lower bound of the form $g(r) \geq c2^r$ for some constant $c > 2$. Our main result further improves this bound; we prove that there exists a constant $c > 0$ for which
\[ g(2r + 1, r) > c(2.587)^r. \] (2)

We would like to remark that Dr Zs. Tuza has kindly informed us that Alles and Poljak [1] have also greatly improved the lower bound in (1): they have in fact shown that $g(r) = \max_n g(n, r) > 1.2(2.50)^r$.

We remark that the construction given by Alles and Poljak [1] proves $g(n, r) > 1.2(2.50)^r$ only for $n = 9r/4$, and so one needs a ‘large’ ground set. It is rather pleasing that we can take our ground set in (2) of size $n = 2r + 1$, since that is the minimal possible value for $n$: note that $K_n(n, r)$ is acyclic for $n \leq 2r$. Also, this ‘minimality’ of $n$ easily gives us a considerable improvement on a certain result on clique covers. Let us recall some definitions. A clique cover of a graph $G$ is a family of complete subgraphs of $G$ such that any edge of $G$ is contained in one of the graphs in the family. The minimal cardinality of such a family is the clique covering number of $G$, and is usually denoted $\text{cc}(G)$. Clique coverings were introduced in [8], and have since been studied in many papers; see e.g. [3], [5], [7] and [12].

Let $H^n$ be the complement of an $n$-cycle. It was proved by de Caen, Gregory and Pullman [7] that $\text{cc}(H^n) \leq 2 \log_2(n - 1) + 2$. Alles and Poljak’s bound of $g(9r/4, r) > 1.2(2.50)^r$ improves this to $\text{cc}(H^n) < 1.695 \log_2 n$. Relation (2) easily implies
\[ \log_2 n \text{cc}(H^n) < 2 \log_2 n \log_2 2.587 < 1.459 \log_2 n, \] (3)
for large enough $n$. It is in fact conjectured in [7] that $\text{cc}(H^n) = (1 + o(1)) \log_2 n$, although Alles and Poljak’s bound and (3) seem to be the first upper bounds of the form $c \log_2 n$, with $c < 2$.

We show (2) by a simple inductive construction; we remark however that our proof involves an elementary computer search. In Section 2 we give a complete proof for the somewhat weaker bound $g(2r + 1, r) > c(2.154)^r$, and make some comments on the computing involved in the proof of (2). In the last two sections, we discuss some related problems.

\section*{2. The construction}

Let us define $f(n, r)$ to be the maximum of the orders of induced paths in $K_n(n, r)$. Since an induced cycle of length $\ell$ contains an induced path with $\ell - 1$ vertices, it follows that $g(n, r)$ is at most $f(n, r) + 1$. The following result shows that, on the other hand, $f(2r + 1, r)$ grows about as fast as $g(2r + 1, r)$. We remark that one can check that $f(2r + 1, r)$ is non-decreasing and that $f(n, 2) = 5$ for any $n \geq 5$. Incidentally, up to permutation of the elements of the ground set, there is only one induced path of order 5 in $K_n(n, 2)$, namely, 12, 34, 15, 24, 13, where we have omitted brackets and commas when writing the 2-sets, e.g. $12 = \{1, 2\}$, etc.

**Lemma 1.** For $r \geq 2$, we have $2 + 2f(2r + 1, r) \leq g(2r + 3, r + 1) \leq 1 + f(2r + 3, r + 1)$.

**Proof.** Suppose that $r \geq 2$ and that
\[ P = V_1V_2 \ldots V_t \]
is an induced path in $\text{Kn}([2r+1], r)$ of maximal order: $t = f(2r+1, r) \geq f(5, 2) = 5$. Set $x = 2r+2$ and $y = 2r+3$. Considering the $2t$ sets $V_i^x = V_i \cup \{x\}$ and $V_i^y = V_i \cup \{y\}$, $i = 1, \ldots, t$, we see that $\text{Kn}([2r+3], r+1)$ contains the disjoint union of two paths of order $t$ as an induced subgraph. We now join these two paths through a vertex to obtain an induced path of order $2t+1$. Pick $a \in V_3 \setminus V_1$ and write $U_1$ for the $(r+1)$-set $V_2 \cup \{a\}$. Trivially, $U_1$ meets neither $V_i^x$ nor $V_i^y$. Moreover, it certainly meets $V_i^x$ and $V_i^y$ for $2 \leq i \leq t$, since $V_2 \subset U_1$ meets $V_i = V_i^x \cap V_i^y$ if $i \neq 3$ and $a \in U_1 \cap V_3^x \cap V_3^y$. This gives us the induced path and we now proceed to close a cycle. Choose an arbitrary $b \in V_{t-2} \setminus V_t$ and define $U_2 = V_{t-1} \cup \{b\}$. By similar considerations to the ones concerning $U_1$, we see that $U_2$ meets neither $V_i^x$ nor $V_i^y$ but does meet $V_i^x$ and $V_i^y$ for $1 \leq i \leq t-1$. Finally, we note that $U_1 \cap U_2 \cap V_2 \cap V_{t-1} = \emptyset$ if $t \geq 5$, and this shows that $U_2$ closes the cycle as required.

We shall now bound $f(2r+1, r)$ from below by giving a recursive construction for induced paths in $\text{Kn}([2r+1], r)$. Our construction is given in Lemma 2 below. As usual, we denote the $k$-subsets of a set $X$ by $X^{(k)}$, $k \geq 0$. For $s = 1, 2, \ldots$, let us define $G_s$ to be the bipartite graph whose vertex classes are $[2s]^{(s)}$ and $[2s]^{(s-1)}$, two vertices in different classes being adjacent if and only if they are disjoint. Define $w(s)$ to be the maximum number of vertices in $[2s]^{(s)}$ in an induced path in $G_s$.

**Lemma 2.** For every $r \geq 2$ and $s \geq 1$,

$$f(2(r+s)+1, r+s) \geq \begin{cases} w(s)(f(2r+1, r)+1) - 1 & \text{if } f(2r+1, r) \text{ is odd} \\ w(s)f(2r+1, r) - 1 & \text{if } f(2r+1, r) \text{ is even}. \end{cases}$$

**Proof.** Let $W = A_1B_1A_2B_2 \ldots A_{m-1}B_{m-1}A_m$ be an induced path in $G_s$, where $A_i \in [2s]^{(s)}$, $B_j \in [2s]^{(s-1)}$, $1 \leq i \leq m$, $1 \leq j \leq m-1$, $m \geq 2$, and let $P = V_1V_2 \ldots V_t$ be an induced path in $\text{Kn}([2s+1, \ldots, 2(r+s)+1], r)$, where $r \geq 2$ and $t \geq 5$ is odd. We now construct an induced path in $\text{Kn}([2(r+s)+1], r+s)$ of order $mt+1$.

For a set $A \subset [2s]$, write $A^c$ for $[2s] \setminus A$. Note that, given two disjoint sets $X_1, X_2$, and families of $r_i$-sets $\mathcal{F}_i \subset X_i^{(r_i)}$, where $r_i \geq 1$ ($i = 1, 2$), the graph induced by $\mathcal{F}_1 \cup \mathcal{F}_2 = \{F_1 \cup F_2 : F_i \in \mathcal{F}_i, i = 1, 2\}$ in $\text{Kn}(X_1 \cup X_2, r_1 + r_2)$ is the (categorical) product of the graphs induced by the $\mathcal{F}_i$ in the $\text{Kn}(X_i, r_i)$. We start our construction by setting $\mathcal{A} = \{A_1, A_2^{(1)}, \ldots, A_m, A_m^{(s)}\}$ and noting that $\mathcal{A}$ induces a disjoint union of at least $[m/2]$ edges in $\text{Kn}([2s], s)$. Hence $\mathcal{A} \cup \{V_1, \ldots, V_t\}$ induces a disjoint union of at least $m$ paths of order $t$ in $\text{Kn}([2(r+s)+1], r+s)$ and, as $t$ is odd, the pairs of end-vertices of these paths are $\{V_i \cup A_i, V_i \cup A_i^c\}$ and $\{V_i \cup A_i, V_i \cup A_i^c\}$, $i = 1, \ldots, m$. Set $P_t$ to be the path with end-vertices $\{V_t \cup A_t, V_t \cup A_t^c\}$, $i = 1, \ldots, m$, and let $P \subset [2(r+s)+1]^{(r+s)}$ be the set of $mt$ vertices used by these $m$ paths. We shall, using a method analogous to the one in the proof of Lemma 1, join these $P_i$ together to obtain our path in $\text{Kn}([2(r+s)+1], r+s)$.

Pick $a \in V_3 \setminus V_1$ and $b \in V_{t-2} \setminus V_t$. For $i = 1, \ldots, m-1$, define $U_i = \begin{cases} V_{i-1} \cup B_i \cup \{b\} & \text{if } i \text{ is odd} \\ V_2 \cup B_i \cup \{a\} & \text{if } i \text{ is even}. \end{cases}$
Note that these $U_i$ span an independent set. Indeed, for $1 \leq i < j \leq m - 1$, we have $U_i \cap U_j \supseteq V_2 \cap V_{t-1} \neq \emptyset$, as $t \geq 5$. Furthermore, for $1 \leq i \leq m - 1$, we claim that $U_i$ is adjacent to exactly two vertices in $P$, namely, $V_t \cup A_i$ and $V_t \cup A_{i+1}$ if $i$ is odd and $V_1 \cup A_i$ and $V_1 \cup A_{i+1}$ if $i$ is even. We check this statement only for even $i$, since the other case is analogous. Let $Y$ be a vertex in $P$, $Y = V_t \cup W$, where $W$ is either $A_k$ or $A_{k+1}$ for some $1 \leq k \leq m$. First of all, $U_i \cap V_2$ meets $Y$ if $4 \leq j \leq t$ or $j = 2$, since $V_2 \cap V_j \neq \emptyset$ in that case. Secondly, $a \in U_i \cap Y \neq \emptyset$ if $j = 3$. Finally, if $j = 1$ then, noting that $U_i \cap Y = B_i \cup A_k$, we have that $U_i$ does not meet $Y = V_1 \cup A_k$ if and only if $k = i$ or $k = i + 1$.

**Corollary 3.** There is an absolute constant $c > 0$ for which $g(2r + 1, r) > c10^{r/3}$ holds for every $r \geq 1$.

**Proof.** In view of Lemma 1, it is enough to prove that $f(2r + 1, r) > c010^{r/3}$ for some constant $c_0 > 0$. Let us apply induction on $r$ in order to bound $f(2r + 1, r)$ from below. Recall that $f(5, 2) = 5$. By Lemma 1, we have that $f(7, 3) \geq 11$ and $f(9, 4) \geq 23$. Also,

$$W = 123, 56, 124, 36, 125, 34, 156, 24, 135, 26, 345, 16, 235, 14, 236, 15, 246, 13, 456$$

is an induced path in $G_3$, so that $w(3) \geq 10$. Thus Lemma 2 yields, for $u \geq 0$,

$$f(6u + 5, 3u + 2) = f(2(3u + 2) + 1, 3u + 2) \geq 6 \cdot 10^u - 1,$$

$$f(6u + 7, 3u + 3) = f(2(3u + 3) + 1, 3u + 3) \geq 12 \cdot 10^u - 1$$

and

$$f(6u + 9, 3u + 4) = f(2(3u + 4) + 1, 3u + 4) \geq 24 \cdot 10^u - 1,$$

which implies that indeed $f(2r + 1, r) > c010^{r/3}$ for a positive $c_0$.

**Theorem 4.** There is an absolute constant $c > 0$ for which $g(2r + 1, r) > c300^{r/6}$ holds for every $r \geq 1$.

It is clear from the proof of Corollary 3 that we need only show that $w(6) \geq 300$ in order to prove the lower bound in Theorem 4. Unfortunately, however, the only way we have managed to prove such a bound for $w(6)$ is by a computer search. We do not describe the induced path we have found; we only remark that an elementary depth-first algorithm finds such a path after hitting dead-ends only 66 times, i.e. the 67th maximal path we analyse has 300 vertices in $[12]^{(6)}$.

Our algorithm explores the possible extensions of the currently considered path in a certain natural order. More precisely, let us assume that we have an induced path in $G_s$ ending at an $(s - 1)$-set and we are trying to consider all its possible extensions for further search. Recall that, for $k = 1, 2, \ldots$, the colex order on $N(k)$ is the ordering in which $A$ precedes $B$ if and only if $sup A \Delta B \subseteq B$, where $\Delta$ denotes symmetric difference. The order in which we make our search is simply the colex order on $[2s]^{(s)}$, i.e. we search the smallest possible extension in the colex order first. If, on the other hand, our path ends at an $s$-set, our search follows the opposite of the colex order on $[2s]^{(s-1)}$.

It is very likely that by using better algorithms and more computer time one could somewhat improve the lower bound in Theorem 4. However, to prove $f(r) >
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(4 - o(1))^r one would have to show that \( \sup w(s)^{1/s} = 4 \). Therefore, in order to make real progress, more systematic work needs to be done on estimating the order of \( w(s) \).

3. General induced subgraphs

In this section we discuss a related problem concerning induced subgraphs of Kneser graphs. For a graph \( H \), let us denote its order by \( |H| \) and its maximal and minimal degrees by \( \Delta(H) \) and \( \delta(H) \), respectively. Also, let us set

\[ h(\Delta, r) = \max \{|H| : H \text{ an induced subgraph of } Kn(N, r), \delta(H) \geq 1, \Delta(H) \leq \Delta \}. \]

Our next result estimates \( h(\Delta, r) \) from both sides; the bounds we give are reasonably strong: their quotient is smaller than 4. The upper bound is proved by the linear algebraic method of Frankl [9] and Kalai [13] (see also Alon [2]). It should be noted that, using a graph-theoretic lemma in [3], one can prove the upper bound below by quoting the set-pair system result of Frankl [9] and Kalai [13]. However, our method is more direct.

**Theorem 5.** For \( r, \Delta \geq 1 \),

\[ \max \left\{ \binom{2r}{r}, (\Delta + 1) \binom{2r - 2}{r - 1} \right\} \leq h(\Delta, r) \leq \Delta \binom{2r}{r}. \]

**Proof.** Let \( r, \Delta \geq 1 \) be fixed. We first prove the lower bound. Note that \( Kn(N, r) \) contains as an induced subgraph the disjoint union of \( \binom{2r}{r}/2 \) edges; indeed, simply consider \( Kn(\lfloor 2r \rfloor, r) \subset Kn(N, r) \). Hence \( h(\Delta, r) \geq \binom{2r}{r} \). Moreover, let us write \( n = 2r + \Delta - 1 \) and define

\[ F = \{ F \subset [n] : |F \cap [\Delta + 1]| = 1 \}. \]

Let \( K^{\Delta+1, \Delta+1} \) denote the balanced complete bipartite graph on \( 2(\Delta + 1) \) vertices, and let \( K' \) denote the graph obtained from \( K^{\Delta+1, \Delta+1} \) by the deletion of a perfect matching. It is easily verified that \( F \) induces a disjoint union of \( \binom{2r-2}{r-1}/2 \) copies of \( K' \). Thus \( h(\Delta, r) \geq (\Delta + 1) \binom{2r-2}{r-1}/2 \), which completes the proof of the lower bound.

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Let \( H \subset Kn(N, r) \) be a finite induced subgraph of \( Kn(N, r) \) with vertices \( v_1, \ldots, v_m, m = |H| \), none of them isolated. Let \( \{x_i \in \mathbb{R}^{2r} : i \geq 1 \} \) be a set of points in \( \mathbb{R}^{2r} \) such that for all \( i_1 < \cdots < i_{2r} \) the vectors \( x_{i_1}, \ldots, x_{i_{2r}} \) are linearly independent, and hence \( x_{i_1} \wedge \cdots \wedge x_{i_{2r}} \in \wedge^{2r} \mathbb{R}^{2r} \) is non-zero. For each \( 1 \leq i \leq m \), set \( y_i = \wedge_{j \in e_i} x_j \in \wedge^{2r} \mathbb{R}^{2r} \). Let \( \{e_i : 1 \leq i \leq m\} \) be a fixed basis of \( \mathbb{R}^m \) and define a linear map \( L : \mathbb{R}^m \to \wedge^r \mathbb{R}^{2r} \) by sending \( e_i \) to \( y_i, 1 \leq i \leq m \). Also, let \( \phi : \wedge^{2r} \mathbb{R}^{2r} \to \mathbb{R} \) be an arbitrary isomorphism and set \( \lambda_{ij} = \phi(y_i \wedge y_j), 1 \leq i, j \leq m \). Denote the \( m \) by \( m \) matrix \( (\lambda_{ij}) \) by \( A \).
Note that \( \ker L \subset \ker A \), where \( A \) is regarded as a linear map on \( \mathbb{R}^m \) acting on the column vectors of \( \mathbb{R}^m \) by left-multiplication. Indeed, let \( (\alpha_j)_1^m \in \mathbb{R}^m \) be such that
\[
\sum_{j=1}^m \alpha_j y_j = L \left[ \sum_{j=1}^m \alpha_j e_j \right] = 0.
\]
Then, for each \( 1 \leq i \leq m \), applying \( y_i \wedge - \) followed by \( \phi \), we get \( \sum_{j=1}^m \alpha_j \lambda_{ij} = 0 \). Hence \( \lambda(\alpha_j) = 0 \). Denote the rank of \( A \) by \( \rho(A) \). The following completes the proof.

Claim. \( m/\Delta \leq \rho(A) \leq \binom{2\rho}{\rho} \).

Note that
\[
m = \dim \ker L + \dim \text{im} L
\leq \dim \ker A + \binom{2r}{r}
= m - \rho(A) + \binom{2r}{r},
\]
which gives the upper bound for \( \rho(A) \). The lower bound follows easily from the fact that, in \( A \), each row has at least one and each column at most \( \Delta \) non-zero entries. Indeed, if we assume that the first \( \rho = \rho(A) \) columns of \( A \) span the image of \( A \), then every row of \( A \) must have one of its non-zero entries in those columns and, therefore, the number of rows \( m \) of \( A \) is at most \( \Delta \rho \), since each column has at most \( \Delta \) non-zero entries.

4. Concluding remarks and open problems

By using the set-pair system result of Frankl [9] and Kalai [13], one can show that \( w(s) \leq 1 + \binom{2s-1}{s-1} \), \( s \geq 1 \). Hence the best lower bound for \( g(2r+1, r) \) one can possibly get by our construction (with \( s \) fixed) seems to be exponentially smaller than \( \binom{2r}{r} \). Being unable to decide whether \( g(r) = o \left( \binom{2r}{r} \right) \) holds, we were led to the following weakening of that question.

Problem 6. Are there absolute constants \( \Delta_0 \geq 1 \) and \( \epsilon_0 > 0 \) such that for every \( r \geq 1 \) the graph \( K_n([2r+1], r) \) contains a connected induced subgraph \( H_r \) with \( \Delta(H_r) \leq \Delta_0 \) and \( |H_r| \geq \epsilon_0 \binom{2r+1}{r} \)?

Alles and Poljak [1] have raised the following very interesting problem. Let \( n = n(r) \) be the smallest integer for which \( g(n, r) = g(r) \). The question then is to determine or estimate \( n(r) \). It has not been disproved that \( n(r) = 2r + 1 \).

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