Graphs and Combinatorics (1999) 15 : 341–351

Graphs and Combinatorics

On the Fractional Intersection Number of a Graph

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Abstract. An *intersection representation* of a graph G is a function $f : V(G) \to 2^S$ (where S is any set) with the property that $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \varnothing$. The *size* of the representation is $|S|$. The *intersection number* of G is the smallest size of an intersection representation of G. The intersection number can be expressed as an integer program, and the value of the linear relaxation of that program gives the *fractional intersection number*. This is in consonance with fractional versions of other graph invariants such as matching number, chromatic number, edge chromatic number, etc.

We examine cases where the fractional and ordinary intersection numbers are the same (interval and chordal graphs), as well as cases where they are wildly different (complete multipartite graphs). We find the fractional intersection number of almost all graphs by considering random graphs.

1. Introduction

A graph $G = (V, E)$ is an *intersection graph* if it can be represented as follows: each vertex v corresponds to a set S_v such that $xy \in E(G)$ iff $S_x \cap S_y \neq \varnothing$. It is easy to see [9] that every graph G is an intersection graph by choosing S_v to be the set of edges in G incident to vertex v. Thus to obtain interesting families of graphs, one typically restricts the type of sets allowed in the intersection representation.

Perhaps the best known restriction is to choose only sets that are intervals on a linearly ordered set, such as the real line. The resulting graphs are known as *interval graphs*. Other well-known classes of intersection graphs include circular-arc graphs (the sets S_v are intervals on the unit circle), unit interval graphs (the sets S_v are unit intervals on the real line), chordal graphs (the sets S_v are subtrees of a fixed tree) and other geometric classes where S_v is a box, circle or polygon in n-space (see [7]).

An intersection representation provides a simple means for storing a graph. By storing S_v for each vertex v, one can compute adjacencies only when needed and

* Research supported in part by the National Security Agency

† Research supported in part by DIMACS

avoid storing the entire edge list. Thus it is advantageous to use sets which minimize storage space. For example, in an interval representation, each set S_v is stored by recording its left and right endpoints.

For general intersection representations, the sets S_v can be stored using indicator vectors. Thus the total storage space depends on $\left|\bigcup_{v \in V} S_v\right|$ which we call the *size* of the representation. The *intersection number* of a graph G, denoted $i(G)$, is the smallest size of an intersection representation of G [5]. As noted earlier, $i(G) \leq |E(G)|$.

An alternate formulation of the intersection number is given in the following result [5].

Proposition 1. *The intersection number of a graph G is the size of a smallest covering of the edge set $E(G)$ by cliques.* □

Corollary 2. *The intersection number $i(G)$ of a graph G is the solution to the following integer program.*

$$\min \sum_{C} w(C) \quad \text{sum over all cliques } C, \quad \text{subject to}$$

$$\sum_{C:e \in E(C)} w(C) \geq 1 \quad \text{for each } e \in E(G)$$

$$w(C) \in \{0, 1\} \quad \text{for each clique } C.$$

Proof. Using Proposition 1, $i(G)$ is the size of a minimum clique cover of $E(G)$. Given any clique cover of G, set $w(C) = 1$ if clique C is in the cover, and $w(C) = 0$ otherwise. This gives a feasible solution to the minimization integer program. The objective value of an optimal solution is $i(G)$. □

The *fractional intersection number* of G, denoted $i_f(G)$, is defined to be the optimal solution to the linear program (LP) relaxation of the integer program given in Corollary 2.

Covering Problem (Primal LP)

$$\min \sum_{C} w(C) \quad \text{sum over all cliques } C, \quad \text{subject to}$$

$$\sum_{C:e \in E(C)} w(C) \geq 1 \quad \text{for each } e \in E(G)$$

$$w(C) \geq 0 \quad \text{for each clique } C.$$

Note that the inequality $w(C) \leq 1$ is satisfied automatically for each clique C in an optimal solution.

The dual of the LP given above is a packing problem.

Packing Problem (Dual LP)

$$\max \sum_e w(e) \quad \text{sum over all edges } e, \quad \text{subject to}$$

$$\sum_{e \in E(C)} w(e) \le 1 \quad \text{for each clique } C$$

$$w(e) \ge 0 \quad \text{for each edge } e.$$

Since every edge is part of some clique, the restriction $w(e) \le 1$ is automatically satisfied for any feasible solution.

Both the primal and dual linear programs are feasible, e.g., setting $w(e) = 0$ for every edge and $w(C) = 1$ for every clique give feasible solutions. Thus by the duality theorem of linear programming, there is an optimal solution, and hence $i_f(G)$ is well-defined.

The study of such fractional analogues of standard graph invariants has been quite extensive including fractional chromatic number, edge chromatic number, matching number, etc.; see [11] for an overview.

Since $i(G)$ is the solution to the minimization IP in Corollary 2, and $i_f(G)$ is the solution to its linear relaxation, we know $i_f(G) \le i(G)$ for every graph G. We record this as a proposition.

Proposition 3. *For all graphs G, $i_f(G) \le i(G)$.* □

Given Proposition 3, it is natural to ask, When are $i_f(G)$ and $i(G)$ equal and how large can the gap between them be? In Section 2, we show that $i_f(G) = i(G)$ when G is a chordal graph, and we prove a little more in the case of interval graphs. In Section 3 we show that the difference between i and i_f can be arbitrarily large. Finally, in Section 4, we compute the fractional intersection number of the random graph.

2. Chordal Graphs and Interval Graphs

If G is a triangle-free graph, then the largest cliques in G are edges and thus $i_f(G) = i(G) = |E(G)|$. After triangle-free graphs, it is natural to next consider the class of chordal graphs, because chordal graphs allow triangles but forbid any larger induced cycles. Formally, G is *chordal* if it contains no induced cycle on 4 or more vertices.

Many results about trees are proven by removing a leaf and invoking induction on the smaller remaining tree. In chordal graphs, the role of leaves is played by simplicial vertices. A *simplicial vertex* in a graph G is a vertex whose neighborhood induces a clique in G. We record the following useful result due to Dirac [4].

Theorem 4. *Every chordal graph has a simplicial vertex.* □

Theorem 5. *If G is a chordal graph then $i(G) = i_f(G)$.*

We prove Theorem 5 by using a weighting algorithm. The algorithm assigns $\{0, 1\}$ weights to the edges and the cliques of a chordal graph G so that the sum of the edge weights in any clique is at most 1 and for any particular edge, the sum of the weights of the cliques containing that edge is at least 1. After presenting the algorithm, we prove that it computes both $i_f(G)$ and $i(G)$.

Algorithm: Weighting
Input: A chordal graph G
Initialize: Initialize all edge weights and clique weights to 0. Mark all edges as "uncovered". Set $G_1 = G$.
For $i = 1$ **to** $|V(G)| - 1$ **do:**

1. Find a simplicial vertex v_i in G_i and let C_i be the clique whose vertex set consists of v_i and all of v_i's neighbors in G_i.
2. IF there is an edge of G_i incident to v_i that is marked "uncovered," THEN
   (a) Give the clique C_i weight 1.
   (b) Pick an edge in G_i incident to v_i that is marked "uncovered," and give it weight 1.
   (c) Mark all edges in C_i "covered."
3. Let $G_{i+1} = G_i - v_i$.

End of for loop.
Output: the sum of the weights of the edges of G.

Proof (of Theorem 5).

Note that the graph G_i defined in the weighting algorithm is chordal for each i, since the property of being chordal is hereditary. The first step of the FOR loop is always possible by Theorem 4.

First we show that for any clique C in G,

$$\sum_{e \in E(C)} w(e) \leq 1,$$

where $w(e)$ is the weight of edge e assigned in the algorithm. Suppose that C is a clique in G containing two edges $e, f \in E(C)$ having $w(e) = w(f) = 1$. Without loss of generality, assume that edge e receives its weight of 1 during the jth pass through the FOR loop and that edge f receives its weight during a later iteration. Thus one of e's endpoints is the vertex v_j which is simplicial in G_j. Edges e and f are in a clique together, thus f's endpoints are adjacent to v_j and therefore, $f \in E(C_j)$. So edge f is marked "covered" at the end of the jth pass of the FOR loop. Hence edge f could never be chosen later to receive weight 1, contradicting our assumption.

Next we show that every edge is part of a clique with weight 1, that is,

$$\sum_{C: e \in E(C)} w(C) \geq 1,$$

where $w(C)$ is the weight of clique C assigned in the algorithm. For any edge e, there exists an integer j so that $e \in E(G_j)$ but $e \notin E(G_{j+1})$. Thus one of e's endpoints is the vertex v_j which is simplicial in G_j. If clique C_j is given a weight of 1 in step (2a) of the jth pass of our algorithm, then C_j is our desired clique. Otherwise, it must be the case that all edges incident to v_j (including e) are already marked "covered" at the jth pass. But in order for edge e to have been marked "covered," it must have belonged to another clique (C_k for some $k < j$) which received a weight of 1 at iteration k. Thus in either case, every edge is part of a clique with weight 1.

We have just shown that the weighting algorithm finds feasible solutions to the primal and dual linear programs given in Section 1. Furthermore, the objective functions are equal because each time we give a clique a weight of 1 we give exactly one edge a weight of 1. Therefore the solutions are optimal and

$$i_f(G) = \sum_{e \in E(G)} w(e) = \sum_{C} w(C).$$

Since the weighting algorithm produces a solution to the LP that happens to be *integral*, it is consequently an optimal solution to the IP, thus $i(G) = i_f(G)$. □

Recall that G is an *interval graph* if we can assign to each $v \in V(G)$ a real interval I_v so that $xy \in E(G)$ iff $I_x \cap I_y \neq \emptyset$. It is well-known (see, e.g. [7]) that interval graphs are chordal, thus Theorem 5 applies to interval graphs. However, a stronger result can be proven.

Theorem 6. *If G is an interval graph then $i_f(G) = i(G) =$ the number of maximal cliques in G.*

Proof. Let G be an interval graph containing exactly M maximal cliques. Recall that $i_f(G) \le i(G)$ for all graphs G. Next we show that $i(G) \le M$. Give each maximal clique of G a weight of 1 and each non-maximal clique a weight of 0. Since every edge is part of at least one maximal clique, this provides a feasible solution to the IP in Corollary 2. Our feasible solution has an objective function value of M, thus the optimal solution, $i(G)$, is at most M, i.e., $i(G) \le M$.

It remains to show $M \le i_f(G)$ when G is an interval graph. We assign a weighting to $E(G)$ which we will show to be a feasible solution to the packing problem in Section 1 with an objective function value of M.

Fix an interval representation of G. Denote the interval corresponding to the vertex x by I_x. By the Helly property [8], in interval graphs (see [7]) every maximal clique C corresponds to a real interval

$$[\ell_C, r_C] = \bigcap_{x \in V(C)} I_x \quad \text{where } \ell_C < r_C.$$

The interval $[\ell_C, r_C]$ is either (i) I_v for some $v \in V(C)$, or (ii) $I_x \cap I_y$ for some $x, y \in V(C)$, depending on whether the leftmost right endpoint of vertices in C and the rightmost left endpoint of vertices in C are endpoints of a single interval I_v or of two distinct intervals I_x and I_y.

For each maximal clique C we choose exactly one edge, $e(C) \in E(C)$ to give a weight of 1 as follows. In case (i), choose $e(C)$ to be any edge of C incident to v; in case (ii), choose $e(C) = xy$. We remark that in either case, the intersection of the two intervals representing $e(C)$'s endpoints is the interval $[\ell_C, r_C]$. Assign all other edges a weight of 0.

Next we show that this weighting provides a feasible solution to the packing LP, that is, for every clique C,

$$\sum_{e \in E(C)} w(e) \leq 1$$

where $w(e)$ is the weight assigned above to edge e.

Suppose there were a clique containing two distinct edges e and f with $w(e) = w(f) = 1$. Let C_1 be the maximal clique for which $e = e(C_1)$ and let C_2 be the maximal clique for which $f = e(C_2)$. Let u_i, w_i be the endpoints of the edge e_i (for $i = 1, 2$). Note that $C_1 \neq C_2$ because $e \neq f$.

By the above remark, $I_{u_1} \cap I_{w_1} = [\ell_{C_1}, r_{C_1}]$ and $I_{u_2} \cap I_{w_2} = [\ell_{C_2}, r_{C_2}]$. Since edges e and f are in a clique together, by the Helly property $I_{u_1} \cap I_{w_1} \cap I_{u_2} \cap I_{w_2} \neq \emptyset$, so $[\ell_{C_1}, r_{C_1}] \cap [\ell_{C_2}, r_{C_2}] \neq \emptyset$. This means that all vertices in C_1 are adjacent to all vertices of C_2, contradicting the assumption that C_1 and C_2 are distinct maximal cliques.

Thus our weighting of edges provides a feasible solution to the packing LP. This feasible solution has an objective function value of M. Thus the optimal solution, $i_f(G)$ is at least M, i.e., $i_f(G) \geq M$. □

Note that the stronger conclusion of Theorem 6 does not apply to chordal graphs in general. The graph in Figure 1 is chordal with $i = i_f = 3$, yet it has $M = 4$ maximal cliques.

3. Complete Multipartite Graphs

We noted earlier that triangle-free graphs G have $i(G) = |E(G)|$. The complete bipartite graph $K_{r,s}$ is triangle-free and thus $i(K_{r,s}) = |E(K_{r,s})| = rs$. Our next proposition generalizes this result to complete multipartite graphs.

Proposition 7. *For $n \geq 3$ we have $i_f(K_{x_1,x_2,\ldots,x_n}) = x_{n-1}x_n$ where $x_i \leq x_j$ whenever $i < j$.*

Proof. Let $A_1, A_2, \ldots, A_n$ denote the sets of vertices in each part of $G = K_{x_1,x_2,\ldots,x_n}$, that is, $V(G) = A_1 \cup A_2 \cup \cdots \cup A_n$ where $|A_i| = x_i$ and $uv \in E(G)$ if and only if there exists $i \neq j$ with $u \in A_i$ and $v \in A_j$.

We show $i_f(G) = x_{n-1}x_n$ by producing feasible solutions to the covering and packing LP's both of which yield an objective function value of $x_{n-1}x_n$.

For the covering LP, give each maximal clique a weight of $\dfrac{1}{x_1 x_2 \cdots x_{n-2}}$ and every other clique a weight of 0. Each edge of the graph is in at least $x_1 x_2 \cdots x_{n-2}$ maximal cliques so the weighting is feasible. The objective function value for this feasible solution is

$$\sum_C w(C) = \left(\frac{1}{x_1 x_2 \cdots x_{n-2}}\right)(x_1 x_2 \cdots x_n) = x_{n-1}x_n.$$

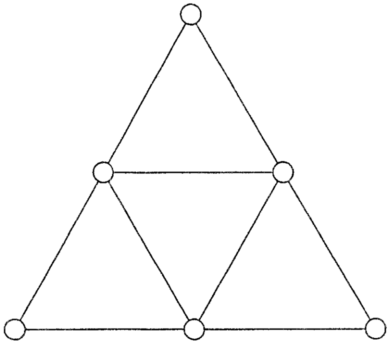

Fig. 1. A chordal graph with $i = i_f = 3$ which has 4 maximal cliques

In the (dual) packing LP give each edge between A_{n-1} and A_n a weight of 1 and every other edge a weight of 0. Since every clique of G contains at most 1 edge between A_{n-1} and A_n, the weighting gives a feasible solution to the packing LP. The objective function value for this solution is

$$\sum_{e \in E(G)} w(e) = 1 \cdot (x_{n-1}x_n) = x_{n-1}x_n.$$

Since the objective function values are equal, we know the solutions are optimal and thus $i_f(G) = x_{n-1}x_n$. □

In the next proposition we compute the intersection number of a specific family of complete multipartite graphs and then note the large gap between i and i_f.

Proposition 8. *Let* $G = K_{2,2,\dots,2}$ *where* $|V(G)| = 2n$. *Then* $i(G) \sim \lg n$.

In particular, we show $\lg n \le i(G) \le \lg n + O(\lg \lg n)$ where "lg" denotes the base-2 logarithm.

Proof. Let $V(G) = \{x_1, y_1, x_2, y_2, \dots, x_n, y_n\}$ where $x_i y_i \notin E(G)$ for $i = 1, 2, \dots, n$. We use the definition of $i(G)$ as the minimum size of a universal set U so that G can be represented as follows: Each vertex $v \in V(G)$ corresponds to a subset $S_v \subseteq U$ so that $vw \in E(G)$ if and only if $S_v \cap S_u \ne \varnothing$.

For the upper bound, we set

$$T = \tfrac{1}{2}\lg n + \tfrac{1}{4}\lg\lg n + J$$

where J is a constant (to be specified below); by properly choosing J we can assume that T is an integer. Let $U = [2T] = \{1, 2, 3, \dots, 2T\}$. Note that the intersection graph on the T-subsets of U is precisely $K_{2,2,2,\dots,2}$ with $\frac{1}{2}\binom{2T}{T}$ parts of size 2. Thus if $\binom{2T}{T} \ge 2n$, we can represent G as an intersection graph with universal set $[2T]$, giving $i(G) \le 2T$. Note that $2T \sim \lg n$, so it remains to show that $\binom{2T}{T} \ge 2n$. To this end, we compute

$$\begin{aligned}\binom{2T}{T} &= \frac{(2T)!}{T!T!} \quad \text{apply Stirling's formula}\\ &\geq K\frac{4^T}{\sqrt{T}} \quad \text{for some constant } K > 0\\ &= K\frac{4^{(1/2)\lg n+(1/4)\lg\lg n+J}}{\sqrt{\frac{1}{2}\lg n+\frac{1}{4}\lg\lg n+J}}\\ &\leq \frac{Kn\sqrt{\lg n}4^J}{\sqrt{\lg n}} \quad \text{provided } n \text{ is sufficiently large}\\ &\geq 2n\end{aligned}$$

where we have chosen J so that $K4^J \geq 2$.

For the lower bound, assume we have a minimum sized universal set U and an appropriate assignment of subsets $S_v \subseteq U$ to vertices $v \in V(G)$. Each subset S_v corresponds to a non-zero indicator vector $(a_1, a_2, \ldots, a_{|U|})$ for which $a_i = 1$ if $i \in S_v$ and $a_i = 0$ otherwise. Vertices in the same part of the multipartite graph G are not adjacent and thus correspond to orthogonal (hence different) vectors. Vertices in different parts get different indicator vectors because they have different neighbor sets. Thus the $2n$ vertices of G have $2n$ different indicator vectors. Hence any representation of G using subsets of U requires that $2^{|U|} \geq 2n$ hence $|U| \geq \lg n$. □

Thus by Propositions 7 and 8 there exist graphs with bounded fractional intersection number but arbitrarily large intersection number.

An anonymous referee noted that if a graph G has n vertices, then $i(G)/i_f(G) = O(\log n)$. The argument is that if a clique C receives weight w in an optimal fractional covering, we choose that clique for an ordinary covering with probability $\min\{1, 3\omega \log n\}$. It follows that the expected number of cliques chosen is $O(i_f(G) \log n)$ and the probabilty that some edge remains uncovered vanishes.

4. Random Graphs

In this section we compute the fractional intersection number of the random graph with edge probability p where p is a constant. We use the standard model of random graphs: Denote by $G_{n,p}$ the graph with n vertices each of whose edges is present with probability p (or absent with probability $1-p$) where each edge is independent of all others.

In [3] and subsequently in [6] the (ordinary) intersection number of random graphs is studied. In [6] it is shown that given a fixed value for p, there exist constants c_1 and c_2 such that, with high probability, the intersection number of $G_{n,p}$

satisfies

$$c_1\left(\frac{n}{\log n}\right)^2 \le i(G_{n,p}) \le c_2\left(\frac{n}{\log n}\right)^2.$$

(The term "with high probability" means that the probability $G_{n,p}$ does not satisfy the above pair of inequalities tends to 0 as $n \to \infty$.)

The situation for fractional intersection number is simpler and we can narrow the constant factor gap between the upper and lower bounds.

Theorem 9. *Let* $0 < p < 1$ *be a constant, and let* $b = 1/p$. *Then, with high probability,*

$$i_f(G_{n,p}) \sim \frac{n^2 p}{4\log_b^2 n}.$$

In particular, we assert that for any fixed $\varepsilon > 0$, with high probability we have

$$(1-\varepsilon)\frac{n^2 p}{4\log_b^2 n} \le i_f(G_{n,p}) \ge (1+\varepsilon)\frac{n^2 p}{4\log_b^2 n}.$$

The key to proving Theorem 9 is understanding the behavior of the clique number of a random graph; see [2] or [10].

Lemma 10. *Let* $0 < p < 1$ *be fixed and let* $b = 1/p$. *Then, with high probability* $\omega(G_{n,p}) \sim 2\log_b n$. □

The essence of the proof of Lemma 10 is as follows. Given n, p, let Q be roughly $2\log_b n$ and let X be the number of Q-cliques in $G_{n,p}$. Observe that

$$E(X) = \binom{n}{Q} p^{\binom{Q}{2}}.$$

Crudely, if $Q \ge (1+\varepsilon)2\log_b n$ one checks that $E(X) \to 0$ as $n \to \infty$ and therefore $P(\omega(G) \ge Q) = P(X > 0) \le E(X) \to 0$, so $\omega(G) < Q$ with high probability.

On the other hand, if $Q \le (1-\varepsilon)2\log_b n$, then $E(X) \to \infty$. Moreover, by computing the variance of X (see [2] or [10]) and applying Chebyshev's inequality we have $P(X = 0) \to 0$ as $n \to \infty$. This gives $\omega(G_{n,p}) \ge (1-\varepsilon)2\log_b n$ with high probability.

(With more care, the value of $\omega(G_{n,p})$ can be pinned down with much greater precision; see [2] or [10].)

For the proof of Theorem 9, we require a stronger result. Let $\varepsilon > 0$ be a fixed number. Applying Janson's inequality (see [1], page 110) one shows that

$$P[X \le (1-\varepsilon)E(X)] = o(n^{-2}). \qquad (*)$$

Indeed, an exponentially small bound on the tail probability can be derived, but for our purposes we only need $o(n^{-2})$.

We are ready to present the proof of Theorem 9.

Proof. Let $\varepsilon > 0$ be a small constant. Let p be fixed and let $b = 1/p$. We know that, with high probability, $\omega = \omega(G_{n,p}) \le (1+\varepsilon)2\log_b n$. We can form a feasible solution to the packing LP formulation for i_f by assigning weight $1/\omega$ to every edge. Thus

$$i_f(G_{n,p}) \ge i(G_{n,p}) \ge \frac{\binom{n}{2}p(1+o(1))}{\binom{\omega}{2}} \ge (1-\varepsilon')\frac{n^2 p(1+o(1))}{4\log_b^2 n}.$$

(This proof of the lower bound is, essentially, the same as that of [3] and [6].)

To prove the upper bound, set $Q = \lfloor (1-\varepsilon)2\log_b n \rfloor$. Let e be any edge of $G_{n,p}$ and let X_e be the number of Q-cliques of $G_{n,p}$ which contain e. Then we have, just as for X, that

$$E(X_e) = \binom{n-2}{Q-2} p^{\binom{Q}{2}-1}.$$

Furthermore, applying the exact same methods used to derive $(*)$ we have that for any $\delta > 0$,

$$P[X_e \le (1-\delta)E(X_e)] = o(n^{-2}).$$

Letting Y denote the number of edges *not* covered by at least $(1-\delta)E(X_e)$ cliques of size Q, we have

$$P(Y > 0) \le E(Y) < n^2 P[X_e \le (1-\delta)E(X_e)] \to 0.$$

Thus, with high probability, all edges of $G_{n,p}$ are covered by at least $(1-\delta)E(X_e)$ cliques of size Q.

In the remarks before this proof, we set X equal to the number of Q-cliques. An application of the second moment method yields $P[X > (1+\delta)E(X)] = o(1)$.

We now specify a fractional edge covering of $G_{n,p}$ by Q-cliques simply by assigning each clique a weight of $W = [(1-\delta)E(X_e)]^{-1}$. This is a feasible fractional covering of the edges of $G_{n,p}$ (every edge is, with high probability, covered by edges of total weight at least 1). Since the probability that $G_{n,p}$ has more than $(1+\delta)E(X)$ Q-cliques is $o(1)$, we have with high probability

$$\begin{aligned} i_f(G_{n,p}) &\le XW \\ &\le \frac{(1+\delta)E(X)}{(1-\delta)E(X_e)} \\ &= (1+\varepsilon')\frac{\binom{n}{Q}p^{\binom{Q}{2}}}{\binom{n-2}{Q-2}p^{\binom{Q}{2}-1}} \\ &= (1+\varepsilon')\frac{n(n-1)p}{Q(Q-1)} \end{aligned}$$

$$\leq (1+\varepsilon'')\frac{n^2 p}{Q^2}$$

$$\leq (1+\varepsilon''')\frac{n^2 p}{4\log_b^2 n}$$

as required. □

Acknowledgments. Many thanks to Noga Alon and Joel Spencer for pointing us to Janson's inequality which gives inequality $(*)$.

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Received: July 1, 1996
Revised: August 11, 1997