

4.4 Likelihood functions:

1. Log likelihood for multinomial responses

Let

$$\mathbf{y}_i = (y_{i1}, y_{i2}, \dots, y_{ik}), \sum_{j=1}^k y_{ij} = m$$

and

$$\boldsymbol{\pi}_i = (\pi_{i1}, \pi_{i2}, \dots, \pi_{ik}), \sum_{j=1}^k \pi_{ij} = 1.$$

Then, the log-likelihood function for observation \mathbf{y}_i is

$$l_i(\boldsymbol{\pi}_i | \mathbf{y}_i) \propto \sum_{j=1}^k y_{ij} \cdot \log(\pi_{ij}) = \sum_{j=1}^{k-1} y_{ij} \cdot \log(\pi_{ij}) + y_{ik} \cdot \log\left(1 - \sum_{j=1}^{k-1} \pi_{ij}\right)$$

If we choose to work with r_{ij} , the log-likelihood function for observation \mathbf{y}_i can be rewritten as

$$\begin{aligned} l_i(\boldsymbol{\pi}_i | \mathbf{y}_i) &\propto \sum_{j=1}^k y_{ij} \cdot \log(r_{ij} - r_{i(j-1)}) \\ &= \sum_{j=1}^{k-1} y_{ij} \cdot \log(r_{ij} - r_{i(j-1)}) + y_{ik} \cdot \log(1 - r_{i(k-1)}) \end{aligned}$$

2. Parameter estimation

For the model with the form

$$\log\left[\frac{r_{ij}(x_i)}{1 - r_{ij}(x_i)}\right] = \theta_j - x_i \beta,$$

we can rewrite the model as

$$\log\left(\frac{r_{ij}}{1 - r_{ij}}\right) = x_{ij}^* \beta^*,$$

where

$$\begin{aligned} r_{ij} &= r_{ij}(x_i), \\ x_{ij}^* &= [0 \quad \cdots \quad 0 \quad \underline{1} \quad 0 \quad \cdots \quad 0 \quad -x_i] \\ &\quad \text{the } j\text{th} \end{aligned}$$

and

$$\beta^* = [\theta_1 \quad \cdots \quad \theta_{k-1} \quad \beta_1 \quad \cdots \quad \beta_p]^t = [\beta_1^* \quad \beta_2^* \quad \cdots \quad \beta_{p+k-1}^*]^t.$$

Then, differentiation with respect to β^* gives

$$\frac{\partial l}{\partial \beta_r^*} = \sum_{i=1}^n \sum_{j=1}^{k-1} \frac{\partial l}{\partial r_{ij}} \frac{\partial r_{ij}}{\partial \beta_r^*} = \sum_{i=1}^n \sum_{j=1}^{k-1} \frac{\partial l}{\partial r_{ij}} [x_{ijr}^* r_{ij} (1 - r_{ij})],$$

where

$$l = \sum_{i=1}^n l_i(\pi_i | \mathbf{y}_i),$$

$$r_{ij} = \frac{\exp(x_{ij}^* \beta^*)}{1 + \exp(x_{ij}^* \beta^*)},$$

and

$$\begin{aligned} \frac{\partial r_{ij}}{\partial \beta_r^*} &= \frac{x_{ijr}^* \exp(x_{ij}^* \beta^*)}{1 + \exp(x_{ij}^* \beta^*)} - \frac{x_{ijr}^* [\exp(x_{ij}^* \beta^*)]^2}{[1 + \exp(x_{ij}^* \beta^*)]^2} \\ &= x_{ijr}^* \left\{ \frac{\exp(x_{ij}^* \beta^*)}{1 + \exp(x_{ij}^* \beta^*)} - \frac{[\exp(x_{ij}^* \beta^*)]^2}{[1 + \exp(x_{ij}^* \beta^*)]^2} \right\} \\ &= x_{ijr}^* (r_{ij} - r_{ij}^2) \\ &= x_{ijr}^* r_{ij} (1 - r_{ij}). \end{aligned}$$

Similarly, the second order derivative can be obtained.