6.2 Noncentral hypergeometric distribution:

There are several ways to derive the univariate noncentral hypergeometric distribution.

Source 1

The noncentral hypergeometric distribution with odd ratio φ is an exponentially weighted version of the central hypergeometric distribution. Let Y has a noncentral hypergeometric distribution. Then,

$$P(Y=y)=\frac{\binom{m_1}{y}\binom{m_2}{s_1-y}\varphi^y}{P_0(\varphi)}=\frac{\binom{m_1}{y}\binom{m_2}{s_1-y}\varphi^y}{\sum_{k=a}^{b}\binom{m_1}{k}\binom{m_2}{s_1-k}\varphi^k},$$

where

$$P_0(\varphi)=\sum_{k=a}^{b}\binom{m_1}{k}\binom{m_2}{s_1-k}\varphi^k$$

and

$$a=max(0,s_1-m_2)\le y\le min(m_1,s_1)=b.$$

The distribution arises in the exponentially weighted sampling scheme in which each of the $\binom{m_1+m_2}{m_1}$ possible samples is weighted proportionally to φ^y. We denote

$$Y\sim H(m,s,\varphi)\ \ or\ \ Y\sim H(s,m,\varphi),$$

where

$$m=(m_1,m_2),s=(s_1,s_2),s_1+s_2=m_1+m_2.$$

Note that

$$H(m,s,1)=H(m,s)$$

is the central hypergeometric distribution.

Source 2:

Let $Y_1\sim B(m_1,\pi_1),Y_2\sim B(m_2,\pi_2)$ are independent and

$$\varphi=\frac{\left(\frac{\pi_1}{1-\pi_1}\right)}{\left(\frac{\pi_2}{1-\pi_2}\right)}.$$

Then, the conditional distribution of Y_1 given $Y_1+Y_2=s_1$ is the noncentral hypergeometric distribution $H(m,s,\varphi)$.

The distribution of $Y \sim H(m, s, \varphi)$ is

$$f(y|\varphi) \propto exp\left\{log\left[\frac{\varphi^y}{P_0(\varphi)}\right]\right\} = exp\{y \cdot log(\varphi) - log[P_0(\varphi)]\}.$$