Supplement 1: Exponential family

Canonical Exponential Family:

The exponential family has the density in the canonical form,

$$f(x_1, x_2, \cdots, x_p | \eta_1, \eta_2, \cdots, \eta_s) = f(x|\eta) = exp\left[\sum_{j=1}^{s} \eta_j T_j(x) - A(\eta)\right] h(x).$$

Note:

Gamma, beta, binomial, Poisson, negative binomial, geometric, and normal distributions are all in the exponential family.

Example 1 (normal distribution):

$X \sim N(\mu, \sigma^2)$**. Then**

$$\begin{aligned} f(x|\mu, \sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] \\ &= exp\left\{\frac{\mu x}{\sigma^2} - \frac{x^2}{2\sigma^2} - \frac{1}{2}\left[{\mu^2}/{\sigma^2} + log(2\pi\sigma^2)\right]\right\}. \end{aligned}$$

Therefore

$$\eta_1 = \frac{\mu}{\sigma^2}, T_1(x) = x, \eta_2 = \frac{-1}{2\sigma^2}, T_2(x) = x^2, A(\eta) = \frac{1}{2}\left[{\mu^2}/{\sigma^2} + log(2\pi\sigma^2)\right].$$

Important result 1:

$$E[T_j(X)] = \frac{\partial A(\eta)}{\partial \eta_j}$$

and

$$Cov[T_j(X), T_k(X)] = \frac{\partial^2 A(\eta)}{\partial \eta_j \partial \eta_k}$$

for $j, k = 1, \cdots, s$.

Example 2 (binomial distribution):

$X \sim B(n, p)$**. Then,**

$$\begin{aligned} f(x|p) &= \binom{n}{x} p^x (1-p)^{n-x} \\ &= exp\left[x \cdot log\left(\frac{p}{1-p}\right) + (n-x) \cdot log(1-p)\right] \cdot \binom{n}{x}. \end{aligned}$$

Therefore

$$\eta_1 = log\left(\frac{p}{1-p}\right), T_1(x) = x, A(\eta_1) = -n \cdot log(1-p) = n \cdot log[1 + exp(\eta_1)].$$

Hence

$$E[T_1(X)] = E(X) = \frac{\partial A(\eta_1)}{\partial \eta_1} = \frac{n \cdot exp(\eta_1)}{1 + exp(\eta_1)} = n \cdot \frac{\left(\frac{p}{1-p}\right)}{\left(\frac{1}{1-p}\right)} = np$$

and

$$Var[T_1(X)] = Cov[T_1(X), T_1(X)] = \frac{\partial^2 A(\eta_1)}{\partial \eta_1^2}$$
$$= \frac{n \cdot exp(\eta_1)}{1 + exp(\eta_1)} - \frac{n \cdot exp(\eta_1) \cdot exp(\eta_1)}{[1 + exp(\eta_1)]^2} = np - np^2 = np(1-p).$$

Important result 2:

The moment generating function for

$$T(X) = [T_1(X) \quad \cdots \quad T_s(X)]$$

Is

$$M_T(t) = M(t_1, \cdots, t_s) = E\{exp[t_1 T_1(X) + \cdots + t_s T_s(X)]\}$$
$$= exp[A(\eta + t) - A(\eta)]$$

and the cumulant generating function is

$$\kappa_T(t) = log[M_T(t)] = A(\eta + t) - A(\eta).$$

Example 3 (Poisson distribution):

$Y \sim P(\lambda)$**. Then,**

$$f(x|\lambda) = \frac{exp(-\lambda)\lambda^x}{x!} = exp[x \cdot log(\lambda) - \lambda] \cdot \frac{1}{x!}.$$

Therefore,

$$\eta_1 = log(\lambda), T_1(x) = x, A(\eta_1) = \lambda = exp(\eta_1).$$

The moment generating function of $T_1(x) = x$ **is**

$$M_T(t) = exp[A(\eta + t) - A(\eta)] = exp[exp(\eta_1 + t) - exp(\eta_1)]$$
$$= exp\{exp(\eta_1)[exp(t) - 1]\} = exp\{\lambda[exp(t) - 1]\}$$

and the cumulant generating function is

$$\kappa_T(t) = \lambda[exp(t) - 1].$$

Note that

$$\kappa_T'(0) = [\lambda \cdot exp(t)]_{t=0} = \lambda = E(X)$$

And

$$\kappa_T''(0) = [\lambda \cdot exp(t)]_{t=0} = \lambda = Var(X).$$

Important result 3:

$T(X) = [T_1(X) \quad \cdots \quad T_s(X)]$ **is distributed according to an exponential family** with density

$$f(t_1, t_2, \cdots, t_s | \eta_1, \eta_2, \cdots, \eta_s) = f(t|\eta) = exp\left[\sum_{j=1}^{s} \eta_j t_j - A(\eta)\right] h^*(t).$$

Important Result 4:

Let X has the density in the canonical form of the exponential family,

$$f(x_1, x_2, \cdots, x_p | \eta_1, \eta_2, \cdots, \eta_s) = f(x|\eta) = exp\left[\sum_{j=1}^{s} \eta_j T_j(x) - A(\eta)\right] h(x), \eta \in \Omega.$$

$T(X) = [T_1(X) \quad \cdots \quad T_s(X)]$ is a complete sufficient statistic for η provided that the exponential family is full rank, i.e.,

(a) neither the $T_j(X)$ nor the η_j satisfy a linear constraint;

(b) Ω contains a s-dimensional rectangle.

Example 1 (normal distribution, continue):

Let the independent random variables $X_1, X_2, \cdots, X_n \sim N(\mu, \sigma^2)$.Then, $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is the complete sufficient statistic for (μ, σ^2).

Example 2 (binomial distribution, continue):

Let the independent random variables $X_1, X_2, \cdots, X_m \sim B(n, p)$. Then, $\sum_{i=1}^{n} X_i$ is the complete sufficient statistic for p.

Example 3 (Poisson distribution, continue):

Let the independent random variables $X_1, X_2, \cdots, X_n \sim P(\lambda)$. Then, $\sum_{i=1}^{n} X_i$ is the complete sufficient statistic for λ.