**Gauss-Jordan reduction:**

**Step 1:** Form the augmented matrix corresponding to the system of linear equations.

**Step 2:** Transform the augmented matrix to the matrix in reduced row echelon form via elementary row operations.

**Step 3:** Solve the linear system corresponding to the matrix in reduced row echelon form. The solution(s) are also for the system of linear equations in step 1.

**Example:**

Solve for the following linear system:

\[
\begin{align*}
 x_1 + x_2 + 2x_3 - 5x_4 &= 3 \\
 2x_1 + 5x_2 - x_3 - 9x_4 &= -3 \\
 2x_1 + x_2 - x_3 + 3x_4 &= -11 \\
 x_1 - 3x_2 + 2x_3 + 7x_4 &= -5
\end{align*}
\]

[solution:]

The Gauss-Jordan reduction is as follows:

**Step 1:**

The augmented matrix is

\[
\begin{bmatrix}
 1 & 1 & 2 & -5 & 3 \\
 2 & 5 & -1 & -9 & -3 \\
 2 & 1 & -1 & 3 & -11 \\
 1 & -3 & 2 & 7 & -5
\end{bmatrix}
\]

**Step 2:**

After elementary row operations, the matrix in reduced row echelon form is
\[
\begin{bmatrix}
1 & 0 & 0 & 2 & -5 \\
0 & 1 & 0 & -3 & 2 \\
0 & 0 & 1 & -2 & 3 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

Step 3:
The linear system corresponding to the matrix in reduced row echelon form is
\[
\begin{align*}
\begin{aligned}
x_1 + 2x_4 &= -5 \\
x_2 - 3x_4 &= 2 \\
x_3 - 2x_4 &= 3
\end{aligned}
\]
The solutions are
\[
x_1 = -5 - 2t, \quad x_2 = 2 + 3t, \quad x_3 = 3 + 2t, \quad x_4 = t, \quad t \in \mathbb{R}
\]
\[
\iff \quad x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
-5 - 2t \\
2 + 3t \\
3 + 2t \\
t \\
\end{bmatrix} = \begin{bmatrix}
-5 \\
2 \\
3 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
-2 \\
3 \\
2 \\
1 \\
\end{bmatrix}t
\]

**Number of solutions of a system of linear equations:**

For any system of linear equations, precisely one of the following is true.
I. The system has exactly one solution.
II. The system has an infinite number of solutions.
III. The system has no solution.

**Note:** the linear system with at least one solution is called consistent and the linear system with no solution is called inconsistent.
Example:

I. Exactly one solution:
Solve for the following system:
\[ \begin{align*}
    x_1 + 2x_2 + 3x_3 &= 9 \\
    2x_1 - x_2 + x_3 &= 8 \\
    3x_1 - x_3 &= 3
\end{align*} \]

[solution:]

The Gauss-Jordan reduction is as follows:

Step 1:
The augmented matrix is
\[
\begin{bmatrix}
    1 & 2 & 3 & 9 \\
    2 & -1 & 1 & 8 \\
    3 & 0 & -1 & 3
\end{bmatrix}.
\]

Step 2:
The matrix in reduced row echelon form is
\[
\begin{bmatrix}
    1 & 0 & 0 & 2 \\
    0 & 1 & 0 & -1 \\
    0 & 0 & 1 & 3
\end{bmatrix}
\]

Step 3:
The solution is
\[ x_1 = 2, \ x_2 = -1, \ x_3 = 3 \]
II. **Infinite number of solutions:**

Solve for the following system:

\[
\begin{align*}
2x_1 + 4x_2 - 2x_3 &= 0 \\
3x_1 + 5x_2 &= 1
\end{align*}
\]

[solution:]
The Gauss-Jordan reduction is as follows:

**Step 1:**
The augmented matrix is

\[
\begin{bmatrix}
2 & 4 & -2 & 0 \\
3 & 5 & 0 & 1
\end{bmatrix}
\]

**Step 2:**
The matrix in reduced row echelon form is

\[
\begin{bmatrix}
1 & 0 & 5 & 2 \\
0 & 1 & -3 & -1
\end{bmatrix}
\]

**Step 3:**
The linear system corresponding to the matrix in reduced row echelon form is

\[
\begin{align*}
x_1 + 5x_3 &= 2 \\
x_2 - 3x_3 &= -1
\end{align*}
\]

The solutions are

\[
x_1 = 2 - 5t, \quad x_2 = -1 + 3t, \quad x_3 = t, \quad t \in \mathbb{R}
\]

\[
\begin{align*}
x &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 - 5t \\ -1 + 3t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix} t
\end{align*}
\]
III. **No solution:**

Solve for the following system:

\[
\begin{align*}
    x_1 + 2x_2 + 2x_3 + 4x_4 &= 5 \\
    x_1 + 3x_2 + 5x_3 + 7x_4 &= 1 \\
    x_1 - x_3 - 2x_4 &= -6
\end{align*}
\]

[solution:]

The Gauss-Jordan reduction is as follows:

**Step 1:**
The augmented matrix is

\[
\begin{bmatrix}
    1 & 2 & 3 & 4 & 5 \\
    1 & 3 & 5 & 7 & 11 \\
    1 & 0 & -1 & -2 & -6
\end{bmatrix}
\]

**Step 2:**
The matrix in reduced row echelon form is

\[
\begin{bmatrix}
    1 & 0 & -1 & -2 & 0 \\
    0 & 1 & 2 & 3 & 0 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

**Step 3:**
The linear system corresponding to the matrix in reduced row echelon form is

\[
\begin{align*}
    x_1 - x_3 - 2x_4 &= 0 \\
    x_2 + 2x_3 + 3x_4 &= 0 \\
    0 &= 1
\end{align*}
\]

Since $0 \neq 1$, there is no solution.