10.5. Analysis of variance and F test:

$F(v_1, v_2)$: F distribution with degrees of v_1 and v_2.

Example 1

$$P(F(3,4) > x) = 0.05 \Rightarrow x = 6.59$$
$$P(F(9,11) > x) = 0.05 \Rightarrow x = 2.90$$
$$P(F(3,4) > x) = 0.01 \Rightarrow x = 16.69$$
$$P(F(3,4) > x) = 0.025 \Rightarrow x = 9.98$$

F test:

As H_0 is true, the random variable with sample value f is $F(k-1, n_T - k)$.

The hypothesis is:

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k \ vs \ H_a: not \ all \ population \ means \ are \ equal.$$

Then,

$$reject \ H_0: \ f > f_{k-1, n_T-k, \alpha}$$
$$not \ reject \ H_0: \ f \leq f_{k-1, n_T-k, \alpha}$$

where

$$f = \frac{MSB}{MSW} = \frac{\left(\sum_{j=1}^{k} n_j (\bar{x}_j - \bar{\bar{x}})^2 \Big/ k-1 \right)}{\left(\sum_{j=1}^{k} (n_j - 1) s_j^2 \Big/ n_T - k \right)}$$

and $f_{k-1, n_T-k, \alpha}$ can be obtained by

$$P\left(F(k-1, n_T - k) > f_{k-1, n_T-k, \alpha}\right) = \alpha.$$

In addition,

$$p - value = P(F(k-1, n_T - k) > f).$$

ANOVA table:

Source	Sum of Squares	Degree of Freedom	Mean Square	F
Between	SSB	$k-1$	MSB	$f = \dfrac{MSB}{MSW}$
Within	SSW	$n_T - k$	MSW	
Total	SST	$n_T - 1$		

12.4: Motivating Example (continue):

Since $k = 3, n_T = 18, \alpha = 0.05$,

$$f = \frac{ns_{\bar{X}}^2}{\left(\frac{\sum_{j=1}^{3}(n_j - 1)s_j^2}{n_T - 3}\right)} = \frac{258}{28.67} = 9 > 3.68 = f_{2,15,0.05} = f_{k-1,n_T-k,\alpha}$$

we reject H_0, where

$$P(F(2,15) > f_{2,15,0.05}) = 0.05 \Rightarrow f_{2,15,0.05} = 3.68.$$

In addition,

$$p - value = P(F(k-1, n_T - k) > f) = P(F(2,15) > 9) = 0.0027 < 0.05$$

we also reject H_0.

Example2:

The following data are the number of products of 3 different production lines.

Number of Products						
Line 1	210	215	205	180	175	190
Line 2	180	160	195	190	170	155
Line 3	145	170	165	160	155	175

Let μ_1, μ_2 and μ_3 be the mean number of products of the 3 production lines. Please test the hypothesis $H_0: \mu_1 = \mu_2 = \mu_3$ at $\alpha = 0.05$.

[Solution:]

$$k = 3, \alpha = 0.05, n_1 = n_2 = n_3 = 6, n_T = n_1 + n_2 + n_3 = 18$$

and

$$\bar{x}_1 = 195.83, \bar{x}_2 = 175, \bar{x}_3 = 161.6, \bar{x} = 177.5$$

Then,

$$MSB = \frac{\sum_{j=1}^{k} n_j(\bar{x}_j - \bar{x})^2}{k-1}$$

$$= \frac{6 \cdot (195.83 - 177.5)^2 + 6 \cdot (175 - 177.5)^2 + 6 \cdot (161.6 - 177.5)^2}{3-1}$$

$$= 1779.4$$

and

$$MSW = \frac{\sum_{j=1}^{k}\sum_{i=1}^{n_j}(x_{j,i} - \bar{x}_j)^2}{n_T - k}$$

$$= \frac{(210 - 195.83)^2 + \cdots + (180 - 175)^2 + \cdots + (175 - 161.6)^2}{18 - 3}$$

$$= 216.91$$

Therefore,

$$f = \frac{MSB}{MSW} = \frac{1779.4}{216.91} = 8.2 > 3.68 = f_{2,15,0.05} = f_{k-1,n_T-k,\alpha}$$

we reject H_0.

Example 3:

The following ANOVA table for 4 normal populations with the same variance σ^2 and means μ_1, μ_2, μ_3 and μ_4.

Source	Sum of Squares	Degree of Freedom	Mean Square	F
Between	(1)	(3)	237.4	(6)
Within	(2)	(4)	(5)	
Total	1909.2	22		

(a) Please complete the above ANOVA table.

(b) Test $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4$ at $\alpha = 0.05$.

[Solution:]

(a)

$$k = 4, (3) = k - 1 = 3,$$

$$(4) = 22 - (3) = 22 - 3 = 19.$$

$$237.4 = \frac{(1)}{(3)} = \frac{(1)}{3} \Longrightarrow (1) = 237.4 \cdot 3 = 712.2.$$

$$(2) = 1909.2 - (1) = 1909.2 - 712.2 = 1197.$$

$$(5) = \frac{(2)}{(4)} = \frac{1197}{19} = 67.$$

$$(6) = \frac{237.4}{(5)} = \frac{237.4}{67} = 3.767.$$

Therefore, ANOVA table is

Source	Sum of Square	Degree of Freedom	Mean Square	F
Between	712.2	3	237.4	3.767
Within	1197	19	63	
Total	1909.2	22		

(b)

$$k = 4, n_T = 23, \alpha = 0.05$$

$$f = 3.767 > 3.13 = f_{3,19,0.05} = f_{k-1,n_T-k,\alpha}$$

we reject H_0.