11.1 Type I and Type II Errors

Question:

How to find a sensible statistical procedure to test if H_0 or H_a is true?

Answer:

A sensible statistical procedure is to make the probability of making a wrong decision as small as possible.

Type I and Type II Errors:

	H_0 is true	H_a is true
Not reject H_0	Correct	Type II error
Reject H_0	Type I error	Correct

Type I error: ***wrong rejection of*** H_0 ***(***H_0 ***is true but is rejected).***

Type II error: ***wrong rejection of*** H_a ***(***H_a ***is true but is rejected).***

$\Rightarrow$ **A sensible statistical procedure should result in small chance of making both type I and type II errors.**

However, it is impossible to have as small chance of making both types of errors simultaneously. The larger the chance of making a Type I error is, the smaller the chance of making a Type II error is.

Let

$\alpha =$ **the probability of making a Type I error.**

$\beta =$ **the probability of making a Type II error.**

$\Rightarrow$ **The larger** α **is, the smaller** β **is.**

Example (continue):

The null hypothesis and alternative hypothesis for the coffee problem

$$H_0 : \mu \geq 3 \text{ vs. } H_a : \mu < 3.$$

Statistical procedure 1:

Throw a dice. Then,

Reject H_0 **: the point is smaller or equal to 3.**

Not reject H_0 **: the point is larger than 3.**

Suppose the dice is fair and the random variable ***X*** represent the point of throwing a dice. As H_0 **is true,**

$$\alpha = P(H_0 \text{ is true, but is rejected}) = P(\mu \geq 3, \text{but the point is smaller or equal to } 3)$$

$$= P(X \leq 3) = \frac{3}{6} = \frac{1}{2}$$

As H_a **is true,**

$$\beta = P(H_a \text{ is true, but not rejected } H_0) = P(\mu < 3, \text{but the point is larger than } 3)$$

$$= P(X > 3) = \frac{3}{6} = \frac{1}{2}$$

Statistical procedure 2:

Throw a dice. Then,

Reject H_0 **: the point is 1.**

Not reject H_0 **: the point is larger than 1.**

As H_0 **is true,**

$$\alpha = P(H_0 \text{ is true, but is rejected}) = P(\mu \geq 3, \text{but the point is equal to } 1)$$

$$= P(X = 1) = \frac{1}{6}$$

As H_a **is true,**

$$\beta = P(H_a \text{ is true, but not rejected } H_0) = P(\mu < 3, \text{but the point is larger than } 1)$$

$$= P(X > 1) = \frac{5}{6}$$

Note (very important):

Usually, we control α to some predetermined numerical value, called *the level of significance*. Then, a sensible or good statistical procedure should result in *as small value of β as possible* with α *controlled to the predetermined level of significance.*

Note:

Some commonly used level of significance include 0.1, 0.05, 0.01.

Note:

Since most applications of hypothesis testing control for the probability of making a Type I error, the probability of making a Type II error would not be controlled.

$\Leftrightarrow$

Since the probability of making a Type II error can not be controlled, "not reject H_0" can still have high probability of making a Type II error. Therefore, we do not "accept H_0" since the probability of making an error is still high. We do "not reject H_0" so far because we can not find enough evidence to "reject H_0".

Example 1:

Let X be normal random variable with mean μ and variance 4.

For the following hypothesis test

$$H_0 : \mu = 1 \text{ vs. } H_a : \mu = 2,$$

we reject H_0 as $X \geq 3$. Please calculate

$$\alpha \equiv \text{the probability of making a type I error}.$$

$$\beta \equiv \text{the probability of making a type II error}.$$

[solutions:]

$$\alpha = P(H_0 \text{ is true, but is rejected}) = P(X \geq 3, \mu = 1)$$

$$= P\left(\frac{X-1}{2} \geq \frac{3-1}{2}\right) = P(Z \geq 1) (\because X \sim N(1,4))$$

$$= 1 - 0.8413 = 0.1587$$

$$\beta = P(H_a \text{ is true, but not reject } H_0) = P(X < 3, \mu = 2)$$
$$= P\left(\frac{X-2}{2} < \frac{3-2}{2}\right) = P(Z < 0.5)\,(\because X \sim N(2,4))$$
$$= 0.6915$$

Exercise:

Let X be binomial random variable with

$$n = 3 \text{ (i.e., } f(i) = \binom{3}{i} p^i (1-p)^{3-i}, i = 0, 1, 2, 3.)\ .$$

For the following hypothesis test

$$H_0 : p = \frac{1}{2} \text{ vs } H_a : p = \frac{2}{3},$$

we reject H_0 as $X = 0$ or 3. Please calculate

$$\alpha \equiv \text{the probability of making a type I error}$$
$$\beta \equiv \text{the probability of making a type II error}\,.$$

Online Exercise:

Exercise 11.1.1