11.1 Type I and Type II Errors

Question:

How to find a sensible statistical procedure to test if $H_0$ or $H_a$ is true?

Answer:

A sensible statistical procedure is to make the probability of making a wrong decision as small as possible.

Type I and Type II Errors:

<table>
<thead>
<tr>
<th></th>
<th>$H_0$ is true</th>
<th>$H_a$ is true</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not reject $H_0$</td>
<td>Correct</td>
<td>Type II error</td>
</tr>
<tr>
<td>Reject $H_0$</td>
<td>Type I error</td>
<td>Correct</td>
</tr>
</tbody>
</table>

Type I error: *wrong rejection of $H_0$ ($H_0$ is true but is rejected).

Type II error: *wrong rejection of $H_a$ ($H_a$ is true but is rejected).

$\Rightarrow$ A sensible statistical procedure should result in small chance of making both type I and type II errors.

However, it is impossible to have as small chance of making both types of errors simultaneously. The larger the chance of making a Type I error is, the smaller the chance of making a Type II error is.

Let

$\alpha = \text{the probability of making a Type I error.}$

$\beta = \text{the probability of making a Type II error.}$

$\Rightarrow$ The larger $\alpha$ is, the smaller $\beta$ is.

Example (continue):

The null hypothesis and alternative hypothesis for the coffee problem
\[ H_0 : \mu \geq 3 \quad \text{vs.} \quad H_a : \mu < 3. \]

**Statistical procedure 1:**

Throw a dice. Then,

- **Reject** \( H_0 \): the point is smaller or equal to 3.
- **Not reject** \( H_0 \): the point is larger than 3.

Suppose the dice is fair and the random variable \( X \) represent the point of throwing a dice. As \( H_0 \) is true,

\[ \alpha = P(H_0 \text{ is true, but is rejected}) = P(\mu \geq 3, \text{ but the point is smaller or equal to 3}) \]

\[ = P(X \leq 3) = \frac{3}{6} = \frac{1}{2} \]

As \( H_a \) is true,

\[ \beta = P(H_a \text{ is true, but not rejected } H_0) = P(\mu < 3, \text{ but the point is larger than 3}) \]

\[ = P(X > 3) = \frac{3}{6} = \frac{1}{2} \]

**Statistical procedure 2:**

Throw a dice. Then,

- **Reject** \( H_0 \): the point is 1.
- **Not reject** \( H_0 \): the point is larger than 1.

As \( H_0 \) is true,

\[ \alpha = P(H_0 \text{ is true, but is rejected}) = P(\mu \geq 3, \text{ but the point is equal to 1}) \]

\[ = P(X = 1) = \frac{1}{6} \]

As \( H_a \) is true,

\[ \beta = P(H_a \text{ is true, but not rejected } H_0) = P(\mu < 3, \text{ but the point is larger than 1}) \]

\[ = P(X > 1) = \frac{5}{6} \]

**Note (very important):**

Usually, we control \( \alpha \) to some predetermined numerical value, called the level of significance. Then, a sensible or good statistical procedure should result in as small value of \( \beta \) as possible with \( \alpha \) controlled to the predetermined level of significance.
Note:
Some commonly used level of significance include 0.1, 0.05, 0.01.

Note:
Since most applications of hypothesis testing control for the probability of making a Type I error, the probability of making a Type II error would not be controlled.
⇒
Since the probability of making a Type II error can not be controlled, “not reject $H_0$” can still have high probability of making a Type II error. Therefore, we do not “accept $H_0$” since the probability of making an error is still high. We do “not reject $H_0$” so far because we can not find enough evidence to “reject $H_0$”.

Example 1:
Let $X$ be normal random variable with mean $\mu$ and variance $4$.
For the following hypothesis test

$$H_0 : \mu = 1 \text{ vs. } H_a : \mu = 2,$$

we reject $H_0$ as $X \geq 3$. Please calculate

$\alpha \equiv \text{the probability of making a type I error}.$

$\beta \equiv \text{the probability of making a type II error}.$

[solutions:]

$\alpha = P(H_0 \text{ is true, but is rejected}) = P(X \geq 3, \mu = 1) = P(Z \geq 3) = 1 - 0.8413 = 0.1587$
\[ \beta = P(H_a \text{ is true, but not reject } H_0) = P(X < 3, \mu = 2) \]
\[ = P\left( \frac{X - 2}{\frac{3 - 2}{2}} < \frac{3 - 2}{2} \right) = P(Z < 0.5) \left( : X \sim N(2, 4) \right) \]
\[ = 0.6915 \]

**Exercise:**

Let \( X \) be binomial random variable with

\[ n = 3 \text{ (i.e., } f(i) = \binom{3}{i} \left( p \right)^i \left( 1 - p \right)^{3-i}, i = 0, 1, 2, 3 \text{).} \]

For the following hypothesis test

\[ H_0 : p = \frac{1}{2} \text{ vs } H_a : p = \frac{2}{3}, \]

we reject \( H_0 \) as \( X = 0 \text{ or } 3 \). Please calculate

\[ \alpha \equiv \text{the probability of making a type I error} \]
\[ \beta \equiv \text{the probability of making a type II error} \]

**Online Exercise:**

*Exercise 11.1.1*