11.3 Modern Approach: P-value

Example (continue):

In the coffee example,

$$H_0 : \mu \geq 3 \text{ vs } H_a : \mu < 3,\ \sigma^2 = 0.18^2,\ \alpha = 0.01$$

Suppose $\bar{x} = 2.92$, the p-value in this case is

$$\begin{aligned}\text{p-value} &= \text{the probability of making type I error by rejecting } H_0 \text{ at } \bar{x} = 2.92 \text{ as } \mu = 3 \\ &= P(\bar{X} < \bar{x},\ \mu = 3) = P(\bar{X} < 2.92,\ \mu = 3) \\ &= P\left(\frac{\bar{X} - 3}{0.18/\sqrt{36}} < \frac{2.92 - 3}{0.18/\sqrt{36}}\right) \approx P\left(Z < \frac{-8}{3}\right) = 0.0038\end{aligned}$$

$$\Rightarrow \text{p-value} = 0.0038 < \text{level of significance} = \alpha = 0.01$$

(the error we might make < (the allowable error)

as reject H_0 at $\bar{x} = 2.92$)

The error we might make by rejecting H_0 at $\bar{x} = 2.92$ is ***smaller*** than the allowable error. Therefore, we ***reject*** H_0.

Suppose $\bar{x} = 2.97$,

$$\begin{aligned}\text{p-value} &= \text{the probability of making type I error by rejecting } H_0 \text{ at } \bar{x} = 2.97 \text{ as } \mu = 3 \\ &= P(\bar{X} < \bar{x},\ \mu = 3) = P(\bar{X} < 2.97,\ \mu = 3) \\ &= P\left(\frac{\bar{X} - 3}{0.18/\sqrt{36}} < \frac{2.97 - 3}{0.18/\sqrt{36}}\right) \approx P(Z < -1) = 0.1587\end{aligned}$$

$$\Rightarrow \text{p-value} = 0.1587 > \text{level of significance} = \alpha = 0.01$$

Since the error we might make by rejecting H_0 at $\bar{x} = 2.97$ is ***larger*** than the allowable error. Therefore, we can ***not reject*** H_0.

Note:

The conclusion based on the p-value is the same as the one based on the previous hypothesis testing procedure.

Note:

In the above example, let $\overline{X}$ **be the sample mean and** $\mu_0 = 3$**. Then,**

$$\text{p-value} = P\left(\overline{X} < \bar{x},\ \mu = \mu_0\right) = P\left(\frac{\overline{X} - \mu_0}{\left(\sigma / \sqrt{n}\right)} < \frac{\bar{x} - \mu_0}{\left(\sigma / \sqrt{n}\right)}\right) \approx P\left(Z < \frac{\bar{x} - \mu_0}{\left(\sigma / \sqrt{n}\right)}\right).$$

For example, as

$$\bar{x} = 2.92 \Rightarrow \text{p-value} = P\left(Z < \frac{2.92 - 3}{0.18 / \sqrt{36}}\right) = 0.0038$$

$$\bar{x} = 2.97 \Rightarrow \text{p-value} = P\left(Z < \frac{2.97 - 3}{0.18 / \sqrt{36}}\right) = 0.1587$$

General Case: as $n \geq 30$ **and level of significance** α

(a) As σ **is known, let**

$$z = \frac{\bar{x} - \mu_0}{\sigma_{\overline{X}}} = \frac{\bar{x} - \mu_0}{\left(\sigma / \sqrt{n}\right)}.$$

(I):

$$H_0 : \mu \geq \mu_0 \text{ vs. } H_a : \mu < \mu_0$$

Then,

$$\text{p-value} = P(Z < z)$$

(II):

$$H_0: \mu \leq \mu_0 \text{ vs. } H_a: \mu > \mu_0$$

Then,

$$\text{p-value} = P(Z > z)$$

(b) As σ is unknown,

$$z = \frac{\overline{x} - \mu_0}{s_{\overline{X}}} = \frac{\overline{x} - \mu_0}{\left(s \big/ \sqrt{n} \right)}.$$

(I):

$$H_0: \mu \geq \mu_0 \text{ vs. } H_a: \mu < \mu_0$$

Then,

$$\text{p-value} = P(Z < z)$$

(II):

$$H_0: \mu \leq \mu_0 \text{ vs. } H_a: \mu > \mu_0$$

Then,

$$\text{p-value} = P(Z > z)$$

In (a) and (b), we then compare p-value and the predetermined level of significance α . If $\text{p-value} < \alpha$, then reject H_0 . Otherwise, we do not reject H_0 .

Example 1 (continue):

$$H_0 : \mu \le 600 \text{ vs. } H_a : \mu > 600.$$

Suppose $n = 36, \bar{x} = 605, s = 12$. What is the conclusion based on p-value as $\alpha = 0.05$.

[solutions:]

$\mu_0 = 600$. Then,

$$\text{p - value} = P\left(Z > \frac{\bar{x} - \mu_0}{\left(s / \sqrt{n} \right)} \right) = P\left(Z > \frac{605 - 600}{\left(12 / \sqrt{36} \right)} \right) = P(Z > 2.5) = 0.0062 < 0.05 = \alpha$$

Therefore, we ***reject*** H_0.

Example 2 (continue):

A sample with $n = 50$ provides a sample mean of 36 and sample standard deviation of 12. Consider the following hypothesis test $H_0 : \mu \le 30 \text{ vs. } H_a : \mu > 30$. Find the p-value. What is your conclusion as $\alpha = 0.0001$?

[solution:]

$$p-value = P(Z > z) = P(Z > 3.535) \approx 0.0002.$$

Since

$$p-value \approx 0.0002 > \alpha = 0.0001,$$

we do ***not*** reject H_0.

Online Exercise:

Exercise 11.3.1
Exercise 11.3.2