4.4. Conditional probability:

$A|B$: **Event A given the condition that event B has occurred.**

Example 10 (continue):

$\{2\}|E_1$: Point 2 occurs given that the point is known to be even.

$P(A|B)$: **The conditional probability of A given B (as the event B has occurred, the chance of the event A then occurs!!)**

Formula of the conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

given $P(B) \neq 0$, and

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

given $P(A) \neq 0$.

Example 10 (continue):

$$P(\{2\}|E_1) = \frac{P(E_1 \cap \{2\})}{P(E_1)} = \frac{P(\{2\})}{P(E_1)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Note: $P(A^c \,|B) + P(A \,|B) = 1$

Note: $P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$.

Independent events: **Two events A and B are said to be independent if**

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B).$$

Dependent events: **Two events A and B are said to be dependent if**

$$P(A|B) \neq P(A)$$

or

$$P(B|A) \neq P(B).$$

Note:

$$P(A \cap B) = P(A)P(B)$$

as events A and B are **independent.**

Intuitively, if events A and B are independent, then the chance of event A occurring is the same no matter whether event B has occurred. That is, event A occurring is "independent" of event B occurring. On the other hand, if events A and B are dependent, then the chance of event A occurring given that event B has occurred will be different from the one with event B not occurring.

Example 12:

	Male Police Officer	Female Police Officer
Getting Promotion	288	36
Not Getting Promotion	672	204

A: the event of a police officer getting promotion.
M: the event of a police officer being man.
W: the event of a police officer being woman.
Then,

$$P(A) = \frac{288 + 36}{1200} = 0.27, P(A|M) = \frac{288}{288 + 672} = 0.3,$$

$$P(A|W) = \frac{36}{204 + 36} = 0.15$$

The above result implies the chance of a promotion knowing the candidate being male is twice higher than the one knowing the one being female. In addition, the chance of a promotion knowing the candidate being female (0.15) is much lower than the overall promotion rate (0.27). That is, the promotion event A is "dependent" on the gender event M or W.

$\Rightarrow$ ***A promotion is not independent of the gender.***

Example 13:

The following are the data on the gender and marital status of 200 customers of a company.

	Male	Female
Single	20	30
Married	100	50

(a) What is the probability of finding a single female customer?
(b) What is the probability of finding a married male customer?
(c) If a customer is female, what is the probability that she is single?

(d) What percentage of customers is male?

(e) If a customer is male, what is the probability that he is married?

(f) Are single and male mutually exclusive? Explain.

(g) Is single independent of female? Explain.

[Solution:]

A_1: the customers are single. A_2: the customers are married.

B_1: the customers are male. B_2: the customers are female.

(a)

$$P(A_1 \cap B_2) = \frac{30}{200} = 0.15.$$

(b) $P(A_2 \cap B_1) = {}^{100}/_{200} = 0.5.$

(c) Since

$$P(B_2) = P(A_1 \cap B_2) + P(A_2 \cap B_2) = \frac{30}{200} + \frac{50}{200} = \frac{80}{200},$$

$$P(A_1|B_2) = \frac{P(A_1 \cap B_2)}{P(B_2)} = \frac{{}^{30}/_{200}}{{}^{80}/_{200}} = 0.375.$$

(d)

$$P(B_1) = P(A_1 \cap B_1) + P(A_2 \cap B_1) = \frac{20}{200} + \frac{100}{200} = 0.6.$$

(e)

$$P(A_2|B_1) = \frac{P(A_2 \cap B_1)}{P(B_1)} = \frac{{}^{100}/_{200}}{{}^{120}/_{200}} = \frac{5}{6}.$$

(f) Single and male are not mutually exclusive since $P(A_1 \cap B_1) \neq 0$.

(g) Single and female are not independent since

$$P(A_1|B_2) = \frac{30}{80} \neq \frac{50}{200} = P(A_1).$$

Supplement Example:

You are given the following information on Events A, B, and C

$$P(A) = 0.1, P(B) = 0.3, P(C) = 0.4, P(A \cap C) = 0.04,$$
$$P(B|A) = 0.9, P(C|B) = 0.6.$$

(a) Compute $P(A^c \cap C)$.

(b) Compute $P(C^c \cup B)$.

(c) Are A and B mutually exclusive? Explain.

(d) Are A and C independent? Explain.

[Solution:]

(a) $\boldsymbol{P(A^c \cap C) = P(C) - P(A \cap C) = 0.4 - 0.04 = 0.36}$.

(b) $\boldsymbol{P(C^c \cup B) = 1 - P(B^c \cap C) = 1 - [P(C) - P(B \cap C)] = 1 - (0.4 - 0.18)}$

$\boldsymbol{= 0.78}$

since $\boldsymbol{P(B \cap C) = P(B)P(C|B) = 0.3 \cdot 0.6 = 0.18}$.

(c) Since $\boldsymbol{P(A \cap B) = P(A)P(B|A) = 0.1 \cdot 0.9 = 0.09 \neq 0}$, $\boldsymbol{A}$ **and** $\boldsymbol{B}$ **are not mutually exclusive.**

(d) Since $\boldsymbol{P(A \cap C) = 0.04 = 0.1 \cdot 0.4 = P(A)P(C)}$, $\boldsymbol{A}$ **and** $\boldsymbol{C}$ **are independent.**