

## 7.1. The Binomial Probability Distribution

**Example:**

$X_2$ : representing the number of heads as flipping a fair coin twice.

$H$  : head       $T$  : tail .

$$\begin{array}{ccc}
 & \square & \square \\
 X_2 = 0 & T & T \Rightarrow P(X_2 = 0) = \frac{1}{2} \cdot \frac{1}{2} = \binom{2}{0} \frac{1}{2} \cdot \frac{1}{2} \quad (1 \text{ combination})
 \end{array}$$

$$\begin{array}{ccc}
 & \square & \square \\
 X_2 = 1 & H & T \\
 & T & H \Rightarrow P(X_2 = 1) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{2}{1} \frac{1}{2} \cdot \frac{1}{2} \quad (2 \text{ combinations})
 \end{array}$$

$$\begin{array}{ccc}
 & \square & \square \\
 X_2 = 2 & H & H \Rightarrow P(X_2 = 2) = \frac{1}{2} \cdot \frac{1}{2} = \binom{2}{2} \frac{1}{2} \cdot \frac{1}{2} \quad (1 \text{ combination})
 \end{array}$$

$$\begin{aligned}
 \Rightarrow P(X_2 = i) &= f_2(i) = \binom{2}{i} \left(\frac{1}{2}\right)^2 \\
 &= (\text{number of combinations}) \cdot (\text{the probability of every combination})
 \end{aligned}$$

,  $i = 0, 1, 2$ .

$X_3$ : representing the number of heads as flipping a fair coin 3 times.

$$\begin{array}{ccc}
 & \square & \square & \square \\
 X_3 = 0 & T & T & T \Rightarrow P(X_3 = 0) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{0} \left(\frac{1}{2}\right)^3 \quad (1 \text{ combination})
 \end{array}$$

□ □ □  
H T T

$$X_3 = 1 \quad \text{T H T} \Rightarrow P(X_3 = 1) = 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{1} \left(\frac{1}{2}\right)^3 \quad (3 \text{ combinations})$$

T T H

□ □ □  
H H T

$$X_3 = 2 \quad \text{H T H} \Rightarrow P(X_3 = 2) = 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{2} \left(\frac{1}{2}\right)^3 \quad (3 \text{ combinations})$$

T H H

□ □ □

$$X_3 = 3 \quad \text{H H H} \Rightarrow P(X_3 = 3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{3} \left(\frac{1}{2}\right)^3 \quad (1 \text{ combination})$$

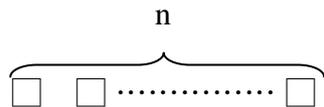
$$\Rightarrow P(X_3 = i) = f_3(i) = \binom{3}{i} \left(\frac{1}{2}\right)^3$$

= (number of combinations) · (the probability of every combination)

,  $i = 0, 1, 2, 3$ .

$X_n$ : representing the number of heads as flipping a fair coin  $n$  times.

Then,



$$X_n = 0 \quad \text{T T} \dots \text{T} \Rightarrow P(X_n = 0) = \left(\frac{1}{2}\right)^n = \binom{n}{0} \left(\frac{1}{2}\right)^n$$

(1 combination)

$$\begin{array}{c}
 \begin{array}{c}
 \overbrace{\square \quad \square \quad \dots \quad \square}^n \\
 \\
 X_n = 1 \quad n \left\{ \begin{array}{l}
 \text{H} \quad \text{T} \dots \dots \text{T} \\
 \text{T} \quad \text{H} \dots \dots \quad \text{T} \\
 \vdots \quad \vdots \quad \ddots \quad \vdots \\
 \text{T} \quad \text{T} \dots \dots \text{H}
 \end{array} \right. \Rightarrow P(X_n = 1) = n \cdot \left(\frac{1}{2}\right)^n = \binom{n}{1} \left(\frac{1}{2}\right)^n \\
 \text{(n combinations)} \\
 \\
 \vdots \\
 \\
 \vdots
 \end{array}
 \end{array}$$

$$\begin{aligned}
 P(X_n = i) &= f_n(i) = \binom{n}{i} \left(\frac{1}{2}\right)^n \\
 &= (\text{number of combinations}) \cdot (\text{the probability of every combination})
 \end{aligned}$$

**Note:** the number of combinations is equivalent to the number of ways as drawing  $i$  balls (heads) from  $n$  balls ( $n$  flips).

**Example:**

$Z_3$ : representing the number of successes over 3 trials.

**S** : Success      **F** : Failure

Suppose the probability of the success is  $\frac{1}{3}$  while the probability of failure is  $\frac{2}{3}$ .

Then,

$$Z_3 = 0 \quad \text{F} \quad \text{F} \quad \text{F} \Rightarrow P(Z_3 = 0) = \binom{1}{3}^0 \binom{2}{3}^3 = \binom{3}{0} \binom{1}{3}^0 \binom{2}{3}^3$$

(1 combination)

S F F

$$Z_3 = 1 \quad \text{F} \quad \text{S} \quad \text{F} \Rightarrow P(Z_3 = 1) = 3 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 = \binom{3}{1} \binom{1}{3} \binom{2}{3}^2$$

(3 combinations)

F F S

S S F

$$Z_3 = 2 \quad \text{S} \quad \text{F} \quad \text{S} \Rightarrow P(Z_3 = 2) = 3 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} = \binom{3}{2} \binom{1}{3}^2 \binom{2}{3}$$

(3 combinations)

F S S

$$Z_3 = 3 \quad \text{S} \quad \text{S} \quad \text{S} \Rightarrow P(Z_3 = 3) = \left(\frac{1}{3}\right)^3 \binom{2}{3}^0 = \binom{3}{3} \binom{1}{3}^3 \binom{2}{3}^0$$

(1 combination)

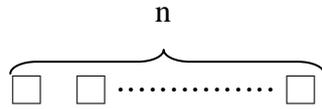
$$\Rightarrow P(Z_3 = i) = f_3(i) = \binom{3}{i} \binom{1}{3}^i \binom{2}{3}^{3-i}$$

= (number of combinations) · (the probability of every combination)

,  $i = 0, 1, 2, 3$ .

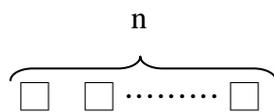
$Z_n$ : representing the number of successes over  $n$  trials.

Then,



$$Z_n = 0 \quad \text{F F} \dots \text{F} \Rightarrow P(Z_n = 0) = \left(\frac{2}{3}\right)^n = \binom{n}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n$$

(1 combination)



$$Z_n = 1 \quad n \left\{ \begin{array}{l} \text{S F} \dots \text{F} \\ \text{F S} \dots \text{F} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ \text{F F} \dots \text{S} \end{array} \right. \Rightarrow P(Z_n = 1) = n \cdot \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^{n-1} = \binom{n}{1} \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^{n-1}$$

(n combinations)

⋮

⋮

$$P(Z_n = i) = f_n(i) = \binom{n}{i} \left(\frac{1}{3}\right)^i \left(\frac{2}{3}\right)^{n-i}$$

= (number of combinations) · (the probability of every combination)

From the above example, we readily describe the binomial experiment.

### Properties of Binomial Experiment

- $X$ : representing the number of successes over  $n$  independent identical trials.
- The probability of a success in a trial is  $p$  while the probability of a failure is  $(1-p)$ .

### Binomial Probability Distribution:

Let  $X$  be the random variable representing the number of successes of a Binomial experiment. Then, the probability distribution function for  $X$  is

$$P(X=i) = f_x(i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i}, \quad i=0,1,2,\dots,n.$$

### Properties of Binomial Probability Distribution:

A random variable  $X$  has the binomial probability distribution  $f(x)$

with parameter  $p$ , then

$$E(X) = np$$

and

$$Var(X) = np(1-p).$$

[Derivation:]

$$\begin{aligned}
E(X) &= \sum_{i=0}^n i \cdot f(i) = \sum_{i=0}^n i \cdot \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^n i \cdot \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^n i \cdot \frac{n!}{i!(n-i)!} p^i (1-p)^{n-i} = \sum_{i=1}^n \frac{n!}{(i-1)!(n-i)!} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^n (np) \cdot \frac{(n-1)!}{(i-1)![(n-1)-(i-1)]!} \cdot p^{i-1} (1-p)^{(n-1)-(i-1)} \\
&= np \sum_{j=0}^{n-1} \frac{(n-1)!}{j![(n-1)-j]!} p^j (1-p)^{n-1-j} \quad (j = i-1) \\
&= np \quad \left( \text{since } \frac{(n-1)!}{j![(n-1)-j]!} p^j (1-p)^{n-1-j} \text{ is the probability} \right. \\
&\quad \left. \text{distribution of a binomial random variable over } n-1 \right. \\
&\quad \left. \text{trials} \right)
\end{aligned}$$

The derivation of  $Var(X) = np(1-p)$  is left as exercise.

### How to obtain the binomial probability distribution:

(a) Using table of Binomial distribution.

(b) Using computer

- by some software, for example, Excel or Minitab.

### Online Exercise:

[Exercise 7.1.1](#)

[Exercise 7.1.2](#)