7.1. The Binomial Probability Distribution

Example:

X_2: representing the number of heads as flipping a fair coin twice.

H : head $\quad T$: tail .

□ □

$X_2 = 0$ T T $\Rightarrow P(X_2 = 0) = \frac{1}{2} \cdot \frac{1}{2} = \binom{2}{0} \frac{1}{2} \cdot \frac{1}{2}$ (1 combination)

□ □

$X_2 = 1$ H T

T H $\Rightarrow P(X_2 = 1) = 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{2}{1} \frac{1}{2} \cdot \frac{1}{2}$ (2 combinations)

□ □

$X_2 = 2$ H H $\Rightarrow P(X_2 = 2) = \frac{1}{2} \cdot \frac{1}{2} = \binom{2}{2} \frac{1}{2} \cdot \frac{1}{2}$ (1 combination)

$$\Rightarrow P(X_2 = i) = f_2(i) = \binom{2}{i} \left(\frac{1}{2} \right)^2$$

$$= (\text{number of combinations}) \cdot (\text{the probability of every combination})$$

, $i = 0, 1, 2.$

X_3: representing the number of heads as flipping a fair coin 3 times.

□ □ □

$X_3 = 0$ T T T $\Rightarrow P(X_3 = 0) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{0} \left(\frac{1}{2} \right)^3$ (1 combination)

□ □ □

H T T

$X_3 = 1$ T H T $\Rightarrow$ $P(X_3 = 1) = 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{1}\left(\frac{1}{2}\right)^3$ (3 combinations)

T T H

□ □ □

H H T

$X_3 = 2$ H T H $\Rightarrow$ $P(X_3 = 2) = 3 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{2}\left(\frac{1}{2}\right)^3$ (3 combinations)

T H H

□ □ □

$X_3 = 3$ H H H $\Rightarrow$ $P(X_3 = 3) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \binom{3}{3}\left(\frac{1}{2}\right)^3$ (1 combination)

$$\Rightarrow P(X_3 = i) = f_3(i) = \binom{3}{i}\left(\frac{1}{2}\right)^3$$

$$= (\text{number of combinations}) \cdot (\text{the probability of every combination})$$

, $i = 0, 1, 2, 3.$

X_n: representing the number of heads as flipping a fair coin *n* times.

Then,

n

□ □ …………… □

$X_n = 0$ T T……………T $\Rightarrow$ $P(X_n = 0) = \left(\frac{1}{2}\right)^n = \binom{n}{0}\left(\frac{1}{2}\right)^n$

(1 combination)

n

□ □ …………… □

$X_n = 1$ n { H T……..……..T

T H……… T $\Rightarrow P(X_n = 1) = n \cdot \left(\frac{1}{2}\right)^n = \binom{n}{1}\left(\frac{1}{2}\right)^n$

⋮ ⋮ ⋱ ⋮ (n combinations)

T T …………..H }

⋮

⋮

$$P(X_n = i) = f_n(i) = \binom{n}{i}\left(\frac{1}{2}\right)^n$$

$= (\text{number of combinations}) \cdot (\text{the probability of every combination})$

Note: the number of combinations is equivalent to the number of ways as drawing i balls (heads) from n balls (n flips).

Example:

Z_3: representing the number of successes over 3 trials.

S : Success F : Failure

Suppose the probability of the success is $\frac{1}{3}$ while the probability of failure is $\frac{2}{3}$.

Then,

□ □ □

$Z_3 = 0$ F F F $\Rightarrow P(Z_3 = 0) = \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^3 = \binom{3}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3$

(1 combination)

□ □ □

S F F

$Z_3 = 1$ F S F $\Rightarrow P(Z_3 = 1) = 3 \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^2 = \binom{3}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$

(3 combinations)

F F S

□ □ □

S S F

$Z_3 = 2$ S F S $\Rightarrow P(Z_3 = 2) = 3 \cdot \left(\frac{1}{3}\right)^2 \cdot \frac{2}{3} = \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)$

(3 combinations)

F S S

□ □ □

$Z_3 = 3$ S S S $\Rightarrow P(Z_3 = 3) = \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^0 = \binom{3}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^0$

(1 combination)

$$\Rightarrow P(Z_3 = i) = f_3(i) = \binom{3}{i}\left(\frac{1}{3}\right)^i\left(\frac{2}{3}\right)^{3-i}$$

$$= (\text{number of combinations}) \cdot (\text{the probability of every combination})$$

, $i = 0, 1, 2, 3.$

Z_n: representing the number of successes over *n* trials.

Then,

$$\overbrace{\square \quad \square \cdots\cdots\cdots\cdots\cdots \square}^{n}$$

$Z_n = 0$ F F……………F $\Rightarrow P(Z_n = 0) = \left(\frac{2}{3}\right)^n = \binom{n}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^n$

(1 combination)

$$\overbrace{\square \quad \square \cdots\cdots\cdots \square}^{n}$$

$Z_n = 1$ n $\left\{\begin{array}{llll} S & F\ldots\ldots\ldots & & .F \\ F & S\ldots\ldots\ldots & & F \\ \vdots & \vdots & \ddots & \vdots \\ F & F & \ldots & .S \end{array}\right.$ $\Rightarrow P(Z_n = 1) = n \cdot \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n-1} = \binom{n}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{n-1}$

(n combinations)

$\vdots$

$\vdots$

$$P(Z_n = i) = f_n(i) = \binom{n}{i}\left(\frac{1}{3}\right)^i\left(\frac{2}{3}\right)^{n-i}$$

$= (\text{number of combinations}) \cdot (\text{the probability of every combination})$

From the above example, we readily describe the binomial experiment.

Properties of Binomial Experiment

- ***X***: **representing the number of successes over *n* independent identical trials.**
- **The probability of a success in a trial is *p* while the probability of a failure is *(1-p)*.**

Binomail Probability Distribution:

Let *X* be the random variable representing the number of successes of a Binomial experiment. Then, the probability distribution function for *X* is

$$P(X=i)=f_x(i)=\binom{n}{i}p^i(1-p)^{n-i}=\frac{n!}{i!(n-i)!}p^i(1-p)^{n-i},\ i=0,1,2,..,n.$$

Properties of Binomial Probability Distribution:

A random variable X has the binomial probability distribution $f(x)$

with parameter p **, then**

$$E(X)=np$$

and

$$Var(X)=np(1-p).$$

[Derivation:]

$$E(X)=\sum_{i=0}^{n} i\cdot f(i)=\sum_{i=0}^{n} i\cdot\binom{n}{i}p^{i}(1-p)^{n-i}=\sum_{i=0}^{n} i\cdot\frac{n!}{i!(n-i)!}p^{i}(1-p)^{n-i}$$

$$=\sum_{i=1}^{n} i\cdot\frac{n!}{i!(n-i)!}p^{i}(1-p)^{n-i}=\sum_{i=1}^{n}\frac{n!}{(i-1)!(n-i)!}p^{i}(1-p)^{n-i}$$

$$=\sum_{i=1}^{n}(np)\cdot\frac{(n-1)!}{(i-1)![(n-1)-(i-1)]!}\cdot p^{i-1}(1-p)^{(n-1)-(i-1)}$$

$$=np\sum_{j=0}^{n-1}\frac{(n-1)!}{j![(n-1)-j]!}p^{j}(1-p)^{n-1-j}\quad (j=i-1)$$

$=np$ (since $\frac{(n-1)!}{j![(n-1)-j]!}p^{j}(1-p)^{n-1-j}$ is the probability distribution of a binomial random variable over $n-1$ trials)

The derivation of $Var(X)=np(1-p)$ **is left as exercise.**

How to obtain the binomail probability distribution:

(a) Using table of Binomail distribution.
(b) Using computer

- **by some software, for example, Excel or Minitab.**

Online Exercise:

Exercise 7.1.1
Exercise 7.1.2