9.4. Two-tailed tests about a population mean: Large sample case ($n \geq 30$):

Objective: Find a sensible statistical procedure to test

$$H_0: \mu = \mu_0 \ vs \ H_a: \mu \neq \mu_0.$$

Derivation of a sensible test:

Intuitively, if $H_0: \mu = \mu_0$ is true, $\bar{X} \approx N\left(\mu_0, \left(\sigma / \sqrt{n}\right)^2\right)$. Then, it is very likely that the sample mean $\bar{x}$ should have value close to μ_0. On the other hand, if $\bar{x}$ is much larger or smaller than μ_0, it seems to indicate that $H_a: \mu \neq \mu_0$ is a sensible hypothesis. Thus, a sensible statistical procedure would be

$$reject \ H_0: |\bar{x} - \mu_0| > c$$
$$not \ reject \ H_0: |\bar{x} - \mu_0| \leq c,$$

where c is some constant.

Next Question: How to determine the value of c?

Answer: Control α (the probability of making type I error) to determine the value of c.

Since

$$\mu = \mu_0, \bar{X} \approx N\left(\mu_0, \left(\sigma / \sqrt{n}\right)^2\right)$$

Then,

$$\begin{aligned}
\alpha &= the \ probability \ of \ wrong \ rejection \ of \ H_0 \\
&= P(H_0 \ is \ true \ but \ is \ rejected) = P(\mu = \mu_0, |\bar{X} - \mu_0| > c) \\
&= P\left(\left|\frac{\bar{X} - \mu_0}{\left(\sigma / \sqrt{n}\right)}\right| > \frac{c}{\left(\sigma / \sqrt{n}\right)}\right) \approx P\left(|Z| > \frac{c}{\left(\sigma / \sqrt{n}\right)}\right)
\end{aligned}$$

$$\Rightarrow \frac{c}{\left(\sigma / \sqrt{n}\right)} = z_{\alpha/2} \Rightarrow c = z_{\alpha/2}\left(\sigma / \sqrt{n}\right).$$

Thus,

$$reject \ H_0: |\bar{x} - \mu_0| > z_{\alpha/2}\left(\sigma / \sqrt{n}\right)$$

$$not \ reject \ H_0: |\bar{x} - \mu_0| \leq z_{\alpha/2}\left(\sigma / \sqrt{n}\right)$$

is a sensible statistical procedure. Furthermore, denote. Thus, by dividing ${}^{\sigma}/_{\sqrt{n}}$ on the both sides, the above sensible statistical procedure can be simplified to

$$reject\ \ H_0: |z| = \left|\frac{\bar{x}-\mu_0}{\sigma_{\bar{X}}}\right| = \left|\frac{\bar{x}-\mu_0}{\left({}^{\sigma}/_{\sqrt{n}}\right)}\right| > z_{\alpha/2}$$

$$not\ \ reject\ \ H_0: |z| \le z_{\alpha/2}$$

In addition,

$$\begin{aligned} & p-value \\ &= the\ \ probabiltity\ \ of\ \ making\ \ type\ \ I\ \ error\ \ by\ \ rejecting\ \ H_0 \\ &\quad at\ \ \bar{x}\ \ as\ \ \mu=\mu_0 \\ &= P(\mu=\mu_0, |\bar{X}-\mu_0| > |\bar{x}-\mu_0|) = P\left(\mu=\mu_0, \left|\frac{\bar{X}-\mu_0}{\left({}^{\sigma}/_{\sqrt{n}}\right)}\right| > \left|\frac{\bar{x}-\mu_0}{\left({}^{\sigma}/_{\sqrt{n}}\right)}\right|\right) \\ &\approx P(|Z| > |z|) \end{aligned}$$

General Case: $n \ge 30$ and level of significance α

- As σ is known,

$$z = \frac{\bar{x}-\mu_0}{\sigma_{\bar{X}}} = \frac{\bar{x}-\mu_0}{\left({}^{\sigma}/_{\sqrt{n}}\right)}.$$

- As σ is unknown,

$$z = \frac{\bar{x}-\mu_0}{s_{\bar{X}}} = \frac{\bar{x}-\mu_0}{\left({}^{s}/_{\sqrt{n}}\right)}.$$

3. $H_0: \mu = \mu_0\ \ vs\ \ H_a: \mu \ne \mu_0$:

$$reject\ \ H_0: |z| > z_{\alpha/2}$$

$$not\ \ reject\ \ H_0: |z| \le z_{\alpha/2}$$

In addition,

$$p-value = P(|Z| > |z|)$$

- If $p-value < \alpha$, then reject H_0.
- If $p-value \ge \alpha$, then do not reject H_0.

Example 4:

Objective: check if the produced golf balls have an average distance in carry and roll of 280 yards as $n = 36, \overline{x} = 278.5, \sigma = 12, \alpha = 0.05$.

[Solution:]

$$H_0: \mu = \mu_0 = 280 \ \ vs \ \ H_a: \mu \neq \mu_0 = 280.$$

Then,

$$z = \frac{\overline{x} - \mu_0}{\left(\sigma/\sqrt{n}\right)} = \frac{278.5 - 280}{\left(12/\sqrt{36}\right)} = -0.75.$$

.

Since

$$|z| = |-0.75| = 0.75 < 1.96 = z_{0.025} = z_{\alpha/2},$$

we thus do not reject H_0. In addition,

$$p - value = P(|Z| > |z|) = P(|Z| > 0.75) = 0.4532 > 0.05 = \alpha,$$

we do not reject H_0.

Interval estimation and hypothesis testing

For $H_0: \mu = \mu_0 \ \ vs \ \ H_a: \mu \neq \mu_0$ with σ known, we do not reject H_0 as

$$|z| = \left|\frac{\overline{x} - \mu_0}{\left(\sigma/\sqrt{n}\right)}\right| = \left|\frac{\mu_0 - \overline{x}}{\left(\sigma/\sqrt{n}\right)}\right| \leq z_{\alpha/2}$$

$$\Leftrightarrow -z_{\alpha/2} \leq \frac{\mu_0 - \overline{x}}{\left(\sigma/\sqrt{n}\right)} \leq z_{\alpha/2}$$

$$\Leftrightarrow -z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 - \overline{x} \leq z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\Leftrightarrow \overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\Leftrightarrow \mu_0 \in \left[\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

$$\Leftrightarrow \mu_0 \ \ falls \ \ in \ \ (1-\alpha)\cdot 100\% \ \ C.I. of \ \ \mu$$

On the other hand, we reject H_0 as

$$\Leftrightarrow \mu_0 \notin \left[\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

$$\Leftrightarrow \mu_0 \ \ do \ \ not \ \ falls \ \ in \ \ (1-\alpha)\cdot 100\% \ \ C.I. of \ \ \mu$$

A confidence Interval approach to $H_0: \mu = \mu_0 \ \ vs \ \ H_a: \mu \neq \mu_0$:
$n \geq 30$ and level of significance α

Step 1: Construct a $(1-\alpha)\cdot 100\%$ confidence interval

- As σ is known,

$$\bar{x} \pm z_{\alpha/2}\sigma_{\bar{X}} = \bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = \left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

- As σ is unknown,

$$\bar{x} \pm z_{\alpha/2}s_{\bar{X}} = \bar{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}} = \left[\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}}\right]$$

Step 2: If μ_0 falls into the above confidence intervals, then do not reject H_0.
Otherwise, reject H_0.

Example 4 (continue):

In the previous example,

$$n = 36, \bar{x} = 278.5, \sigma = 12, \alpha = 0.05,$$
$$H_0: \mu = \mu_0 = 280 \ \ vs \ \ H_a: \mu \neq \mu_0 = 280.$$

We can also use confidence interval approach to hypothesis testing. A 95% confidence interval for μ is

$$\bar{x} \pm z_{0.05/2}\frac{\sigma}{\sqrt{n}} = 278.5 \pm 1.96 \cdot \frac{12}{\sqrt{36}} = 278.5 \pm 3.92 = [274.58, 282.42].$$

Since

$$\mu_0 = 280 \in [274.58, 282.42],$$

we do not reject H_0.

Example 5:

The average starting salary of a college graduate is $\$19000$ according to government's report. The average salary of a random sample of 100 graduates is $\$18800$. The standard error is 800.

(a) Is the government's report reliable as the level of significance is 0.05.

(b) Find the p-value and test the hypothesis in (a) with the level of significance $\alpha = 0.01$.

(c) The other report by some institute indicates that the average salary is $\$18900$. Construct a 95% confidence interval and test if this report is reliable.

[Solution:]

(a)

$$n = 100, \bar{x} = 18800, s = 800, \alpha = 0.05,$$
$$H_0: \mu = \mu_0 = 19000 \ \ vs \ \ H_a: \mu \neq \mu_0 = 19000.$$

Then,

$$|z| = \left|\frac{\bar{x}-\mu_0}{\left(s/\sqrt{n}\right)}\right| = \left|\frac{18800-19000}{\left(800/\sqrt{100}\right)}\right| = |-2.5| = 2.5 > 1.96 = z_{0.025} = z_{\alpha/2}.$$

.

Therefore, reject H_0.

(b)

$p-value = P(|Z| > |z|) = P(|Z| > |2.5|) = 2P(Z > 2.5) = 0.0124 > 0.01$,

we do not reject H_0.

(c)

$$H_0: \mu = \mu_0 = 18900 \;\; vs \;\; H_a: \mu \neq \mu_0 = 18900.$$

A 95% confidence interval is

$$\bar{x} \pm z_{0.05/2}\frac{s}{\sqrt{n}} = 18800 \pm 1.96 \cdot \frac{800}{\sqrt{100}} = [18643.2, 18956.8].$$

Since

$$\mu_0 = 18900 \in [18643.2, 18956.8],$$

we do not reject H_0.

Example 6:

A sample of 49 provides a sample mean of 38 and a sample standard deviation of 7. Let $\alpha = 0.05$. Please test the hypothesis

$$H_0: \mu = \mu_0 = 40 \;\; vs \;\; H_a: \mu \neq \mu_0 = 40.$$

based on

(a) classical hypothesis test.

(b) p-value.

(c) confidence interval.

[Solution:]

$$n = 49, \bar{x} = 38, s = 7, \alpha = 0.05, \mu_0 = 40,$$

$$z = \frac{\bar{x}-\mu_0}{\left(s/\sqrt{n}\right)} = \frac{38-40}{\left(7/\sqrt{49}\right)} = -2.$$

(a)

$$|z| = 2 > 1.96 = z_{0.025} = z_{\alpha/2},$$

we reject H_0.

(b)

$$p-value = P(|Z| > |z|) = P(|Z| > 2) = 0.0456 < 0.05 = \alpha,$$

we reject H_0.

(c)

A $(1-\alpha)\cdot 100\% = 95\%$ **confidence interval is**

$$\overline{x} \pm z_{\alpha/2}\frac{s}{\sqrt{n}} = 38 \pm z_{0.025}\frac{7}{\sqrt{49}} = 38 \pm 1.96 = [36.04, 39.96].$$

Since

$$\mu_0 = 40 \notin [36.04, 39.96],$$

we reject H_0**.**