9.5. Tests about a population mean: Small sample case:

Motivating Example:

Objective: Determine whether Heathrow airport provides superior service as

$$n = 12, \bar{x} = 7.75, s = 1.215, \alpha = 0.05.$$

That is, test the hypothesis

$$H_0: \mu \le \mu_0 = 7 \;\; vs \;\; H_a: \mu > \mu_0 = 7.$$

Intuitively, if $H_0: \mu \le 7$ is true, for example, as $\mu = 7$, it is very likely that the sample mean $\bar{x}$ should have value close to 7. Thus, a sensible statistical procedure would be

$$reject \;\; H_0: \; \bar{x} - 7 > c$$

$$not \;\; reject \;\; H_0: \; \bar{x} - 7 \le c,$$

where c is some constant. Note that the above test is equivalent to

$$reject \;\; H_0: \frac{\bar{x} - \mu_0}{\left(s/\sqrt{n} \right)} = \frac{\bar{x} - 7}{\left(s/\sqrt{n} \right)} > c^*$$

$$not \;\; reject \;\; H_0: \frac{\bar{x} - \mu_0}{\left(s/\sqrt{n} \right)} = \frac{\bar{x} - 7}{\left(s/\sqrt{n} \right)} \le c^*,$$

where $c^* = c \big/ \left(s/\sqrt{n} \right)$. Similar to previous section, we can use level of significance to obtain the constant c^*. As H_0 is true, for ease of explanation, suppose

$$\mu = 7, \frac{\bar{X} - \mu}{\left(S/\sqrt{n} \right)} = \frac{\bar{X} - 7}{\left(S/\sqrt{12} \right)} \sim T(n-1) = T(11).$$

Then,

$$\alpha = 0.05 = the \;\; probability \;\; of \;\; wrong \;\; rejection \;\; of \;\; H_0$$

$$= P(H_0 \;\; is \;\; true \;\; but \;\; is \;\; rejected) = P\left(\mu = 7, \frac{\bar{X} - 7}{\left(S/\sqrt{n} \right)} > c^* \right)$$

$$= P(T(11) > c^*)$$

Therefore,

$$c^* = t_{n-1,\alpha} = t_{11,0.05} = 1.796.$$

Denote

$$t = \frac{\bar{x} - \mu_0}{\left(s/\sqrt{n} \right)}.$$

Thus, a sensible statistical procedure would be

$$reject\ \ H_0:\ t = \frac{\bar{x}-7}{\left({s}/{\sqrt{n}}\right)} > t_{n-1,\alpha} = t_{11,0.05} = 1.796$$

$$not\ \ reject\ \ H_0: t = \frac{\bar{x}-7}{\left({s}/{\sqrt{n}}\right)} \le t_{n-1,\alpha} = t_{11,0.05} = 1.796,$$

with

$$\alpha = P(\mu = 7, reject\ \ H_0) = P\left(\mu = 7, \frac{\bar{X}-7}{\left({S}/{\sqrt{n}}\right)} > t_{n-1,\alpha}\right) = 0.05.$$

Thus,

$$t = \frac{\bar{x}-\mu_0}{\left({s}/{\sqrt{n}}\right)} = \frac{7.75-7}{\left({1.215}/{\sqrt{12}}\right)} = 2.14 > 1.796 = t_{11,0.05}.$$

Therefore, we reject H_0. The other approach is to use p-value. p-value is the probability of making a type I error as reject H_0 at $\bar{x}$, i.e.,

$$\begin{aligned} p-value &= the\ \ probabiltity\ \ of\ \ making\ \ type\ \ I\ \ error\ \ by\ \ rejecting\ \ H_0 \\ &\qquad at\ \ \bar{x}\ \ as\ \ \mu = \mu_0 = 7 \\ &= P\left(\mu = \mu_0, \frac{\bar{X}-\mu_0}{\left({S}/{\sqrt{n}}\right)} > \frac{\bar{x}-\mu_0}{\left({s}/{\sqrt{n}}\right)}\right) = P(T(n-1) > t) = P(T(11) > 2.14) \\ &= 0.028 < 0.05 = \alpha \end{aligned}$$

we reject H_0.

General Case: $n < 30$, normal population, and level of significance α

$$t = \frac{\bar{x}-\mu_0}{s_{\bar{X}}} = \frac{\bar{x}-\mu_0}{\left({s}/{\sqrt{n}}\right)}.$$

1. $H_0: \mu \ge \mu_0\ \ vs\ \ H_a: \mu < \mu_0$:

$$reject\ \ H_0:\ t < -t_{n-1,\alpha}$$
$$not\ \ reject\ \ H_0:\ t \ge -t_{n-1,\alpha}$$

In addition,

$$p-value = P(T(n-1) < t).$$

2. $H_0: \mu \le \mu_0 \;\; vs \;\; H_a: \mu > \mu_0$:

$$reject \;\; H_0: \; t > t_{n-1,\alpha}$$

$$not \;\; reject \;\; H_0: \; t \le t_{n-1,\alpha}$$

In addition,

$$p - value = P(T(n-1) > t).$$

3. $H_0: \mu = \mu_0 \;\; vs \;\; H_a: \mu \ne \mu_0$:

$$reject \;\; H_0: |t| > t_{n-1,\alpha/2}$$

$$not \;\; reject \;\; H_0: |t| \le t_{n-1,\alpha/2}$$

In addition,

$$p - value = P(|T(n-1)| > |t|).$$

Confidence interval approach for two-sided test $H_0: \mu = \mu_0 \;\; vs \;\; H_a: \mu \ne \mu_0$:

$$\mu_0 \notin \left[\bar{x} - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right] \Rightarrow reject \;\; H_0.$$

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Note:

For two-sided test, all the above three methods are equivalent!!

Example 7:

Objective: determine if the mean filling weight is exactly 16 ounces.

That is to test

$$H_0: \mu = \mu_0 = 16 \;\; vs \;\; H_a: \mu \ne \mu_0 = 16$$

with the following data

16.02	16.22	15.82	15.92	16.22	16.32	16.12	15.92

Suppose the level of significance is 0.05.

[Solution:]

$$n = 8, \bar{x} = 16.07, s = 0.18, \alpha = 0.05, \mu_0 = 16,$$

Thus,

$$|t| = \left|\frac{\bar{x} - \mu_0}{\left(s/\sqrt{n}\right)}\right| = \left|\frac{16.07 - 16}{\left(0.18/\sqrt{8}\right)}\right| = 1.1 < 2.365 = t_{7,0.025} = t_{n-1,\alpha/2},$$

we do not reject H_0.

Example 8:

A sample of 6 observations, $18, 20, 16, 19, 17, 18,$ is taken. Suppose the population is *normally distributed.* Then,

(a) for the hypothesis, $H_0: \mu \ge 20 \;\; vs \;\; H_a: \mu < 20$, using $\alpha = 0.05$, test the hypothesis based on the classical hypothesis test?

(b) for the hypothesis, $H_0: \mu \leq 18 \;\; vs \;\; H_a: \mu > 18$, using $\alpha = 0.05$, test the hypothesis based on p-value?

(c) for the hypothesis, $H_0: \mu = 17 \;\; vs \;\; H_a: \mu \neq 17$, using $\alpha = 0.05$, test the hypothesis based on the confidence interval approach?

[Solution:]

$$n = 6, \bar{x} = 18, s = \sqrt{2}, \alpha = 0.05.$$

(a)

$$\mu_0 = 20, t = \frac{\bar{x} - \mu_0}{\left(s/\sqrt{n}\right)} = \frac{18 - 20}{\left(\sqrt{2}/6\right)} = -3.464 < -2.015 = -t_{5,0.05} = -t_{n-1,\alpha},$$

we reject H_0.

(b)

$$\mu_0 = 18, t = \frac{\bar{x} - \mu_0}{\left(s/\sqrt{n}\right)} = \frac{18 - 18}{\left(\sqrt{2}/6\right)} = 0$$

$$\Longrightarrow p - value = P(T(n-1) > t) = P(T(5) > 0) = 0.5 > 0.05 = \alpha,$$

we do not reject H_0.

(c) A 95% confidence interval for μ is

$$\bar{x} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}} = 18 \pm t_{5,0.025} \cdot \frac{\sqrt{2}}{\sqrt{6}} = 18 \pm (2.571 \cdot 0.577) = [16.52, 19.48].$$

Since

$$\mu_0 = 17 \in [16.52, 19.48],$$

we do not reject H_0.

Example 9:

Consider the following hypothesis test, $H_0: \mu \leq 50 \;\; vs \;\; H_a: \mu > 50$. Assume a sample of 16 items provides the following test statistics. Use $\alpha = 0.05$, test the hypothesis based on p-value as

(a) $t = 1.055$.

(b) $t = 3.261$.

[Solution:]

$n = 16$.

(a)

$p - value$

$= P(T(n-1) > t) = P(T(15) > 1.055) > P(T(15) > 1.753) = 0.05 = \alpha$

we do not reject H_0.

(b)

$\boldsymbol{p-value}$

$= \boldsymbol{P(T(n-1) > t) = P(T(15) > 3.261) < P(T(15) > 1.753) = 0.05 = \alpha}$

we reject $\boldsymbol{H_0}$.