Final Review

Chapter 7:

- **Compute the standard error and the probability of the sample mean** $\overline{X}$

$$\overline{X} \approx N\left(\mu, \sigma_{\overline{X}}^2\right) \Longrightarrow \frac{\overline{X}-\mu}{\sigma_{\overline{X}}} = \frac{\overline{X}-\mu}{\left(\sigma/\sqrt{n}\right)} \approx Z \sim N(0,1)$$

$$\overline{P} \approx N\left(\mu, \sigma_{\overline{P}}^2\right) \Longrightarrow \frac{\overline{P}-p}{\sigma_{\overline{P}}} = \frac{\overline{P}-p}{\sqrt{\frac{p(1-p)}{n}}} \approx Z \sim N(0,1)$$

Example 1:

Let X **be a discrete random variable with the following probability distribution:**

$$f_X(x) = \frac{x^2}{100}, x = 1, 2, c, \sqrt{70},$$

and $f_X(x) = 0$, **otherwise, where** c **is some constant.**

(a) Compute c.

(b) Compute $P(X \le 7.5)$.

(c) Compute $E(X)$.

[Solution:]

(a)

$$\sum_x f_X(x) = \frac{1}{100} + \frac{4}{100} + \frac{c^2}{100} + \frac{70}{100} = 1 \Leftrightarrow c = \pm 5.$$

(b)

$$P(X \le 7.5) = P(X = 1 \ \ or \ \ X = 2 \ \ or \ \ X = c)$$
$$= P(X=1) + P(X=2) + P(X=c) = f_X(1) + f_X(2) + f_X(c)$$
$$= \frac{1}{100} + \frac{4}{100} + \frac{25}{100}$$
$$= 0.3$$

(c)

$$E(X) = \sum_x x f_X(x) = 1 \cdot f_X(1) + 2 \cdot f_X(2) + c \cdot f_X(c) + \sqrt{70} \cdot f_X\left(\sqrt{70}\right)$$

$$= 1 \cdot \frac{1}{100} + 2 \cdot \frac{4}{100} + c \cdot \frac{c^2}{100} + \sqrt{70} \cdot \frac{70}{100}$$

$$= \begin{cases} \dfrac{67 + 35\sqrt{70}}{50} = 7.2 \ \ if \ \ c = 5 \\ \dfrac{-58 + 35\sqrt{70}}{50} = 4.7 \ \ if \ \ c = -5. \end{cases}$$

Note:

The other solutions are (a) 5 (c) $\left(67+35\sqrt{70}\right)\Big/_{50}=7.2$

Example 2:

Let $X\sim N(6,5^2)$. Compute

(a) $P(6\leq X\leq 12)$.

(b) $P(|X-6|<5)$.

(c) $P(X^2<9)$.

(d) $P(X<0)$.

[Solution:]

Since $X\sim N(6,5^2)$, then

$$\frac{X-6}{5}=Z\sim N(0,1)$$

(a)

$$P(6\leq X\leq 12)=P\left(\frac{6-6}{5}\leq\frac{X-6}{5}\leq\frac{12-6}{5}\right)=P(0\leq Z\leq 1.2)=0.3849.$$

(b)

$$\begin{aligned}P(|X-6|<5)&=P\left(\left|\frac{X-6}{5}\right|<\frac{5}{5}\right)=P(|Z|<1)=P(-1<Z<1)\\&=2P(0<Z<1)=2\cdot 0.3413\\&=0.6826\end{aligned}$$

(c)

$$\begin{aligned}P(X^2<9)&=P(-3<X<3)=P\left(\frac{-3-6}{5}<\frac{X-6}{5}<\frac{3-6}{5}\right)\\&=P(-1.8<Z<-0.6)=P(0.6<Z<1.8)\\&=P(0<Z<1.8)-P(0<Z\leq 0.6)=0.4641-0.2257\\&=0.2384\end{aligned}$$

(d)

$$\begin{aligned}P(X<0)&=P\left(\frac{X-6}{5}<\frac{0-6}{5}\right)=P(Z<-1.2)=P(Z>1.2)\\&=P(Z>0)-P(0<Z\leq 1.2)=0.5-0.3849\\&=0.1151\end{aligned}$$

Example 3:

The time between arrivals of vehicles at a particular intersection follows an exponential probability distribution with a mean of 12 seconds.

(a) What is the probability that the arrival time between vehicles is 24 seconds or less?

(b) What is the probability of 36 or more seconds between vehicle arrivals?

(c) What is the probability that the arrival time between vehicles is between 12 seconds and 18 seconds?

(d) What is the probability that there are 2 arrivals within 1 minutes?

[Solution:]

(a) Let X be the random variable representing the arrival time between vehicles

$$P(X \leq 24) = 1 - e^{-24/12} = 1 - e^{-2}.$$

(b)

$$P(X \geq 36) = e^{-36/12} = e^{-3}.$$

(c)

$$\begin{aligned} P(12 \leq X \leq 18) &= P(0 \leq X \leq 18) - P(0 \leq X < 12) \\ &= \left(1 - e^{-18/12}\right) - \left(1 - e^{-12/12}\right) \\ &= e^{-1} - e^{-1.5} \end{aligned}$$

(d) Let Y be the random variable representing the number of arrivals within 1 minutes. Then Y is a Poisson random variable with mean $\mu = 60/12 = 5$.

Then,

$$P(Y = 2) = \frac{e^{-5}5^2}{2!} = \frac{25e^{-5}}{2} = 12.5e^{-5}.$$

Example 4:

A bank has kept records of the checking balances of its customers and determined that the average daily balance of its customers is $\$300$ with a standard deviation of $\$48$. A random sample of 144 checking accounts is selected.

(a) What is the probability that the sample mean will be more than or equal to $\$306.60$?

(b) What is the probability that the sample mean will be less than or equal to $\$308$?

(c) What is the probability that the sample mean will be between $\$302$ and $\$308$?

(d) What is the probability that the sample mean will be at least $\$296$?

(e) How large of a sample needs to be taken to provide a 0.4015 probability that the sample mean will be between $\$300$ and $\$304$?

[Solution:]

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{48}{\sqrt{144}} = 4, \mu = 300,$$

for (a) to (d).

(a)

$$P(\overline{X} \geq 306.6) = P\left(\frac{\overline{X}-\mu}{\sigma_{\overline{X}}} = \frac{\overline{X}-300}{4} \geq \frac{306.6-300}{4}\right) \approx P(Z \geq 1.65)$$

$$= 0.5 - P(0 \leq Z < 1.65) = 0.5 - 0.4505$$

$$= 0.0495.$$

(b)

$$P(\overline{X} \leq 308) = P\left(\frac{\overline{X}-\mu}{\sigma_{\overline{X}}} = \frac{\overline{X}-300}{4} \leq \frac{308-300}{4}\right) \approx P(Z \leq 2)$$

$$= 0.5 + P(0 \leq Z \leq 2) = 0.5 + 0.4772$$

$$= 0.9772.$$

(c)

$$P(302 \leq \overline{X} \leq 308) = P\left(\frac{302-300}{4} \leq \frac{\overline{X}-\mu}{\sigma_{\overline{X}}} = \frac{\overline{X}-300}{4} \leq \frac{308-300}{4}\right)$$

$$\approx P(0.5 \leq Z \leq 2) = P(0 \leq Z \leq 2) - P(0 \leq Z < 0.5)$$

$$= 0.4772 - 0.1915$$

$$= 0.2857$$

(d)

$$P(\overline{X} \geq 296) = P\left(\frac{\overline{X}-\mu}{\sigma_{\overline{X}}} = \frac{\overline{X}-300}{4} \geq \frac{296-300}{4}\right) \approx P(Z \geq -1)$$

$$= 0.5 + P(-1 \leq Z \leq 0) = 0.5 + 0.3413$$

$$= 0.8413.$$

(e)

$$P(300 \leq \overline{X} \leq 304) = P\left(\frac{300-300}{\sigma_{\overline{X}}} \leq \frac{\overline{X}-\mu}{\sigma_{\overline{X}}} \leq \frac{304-300}{\sigma_{\overline{X}}}\right)$$

$$\approx P\left(0 \leq Z \leq \frac{4}{\sigma_{\overline{X}}}\right) = 0.4015$$

$$\Leftrightarrow \frac{4}{\sigma_{\overline{X}}} = \frac{4}{\left({}^{48}/_{\sqrt{n}}\right)} \approx 1.29 \Leftrightarrow \sqrt{n} \approx \frac{1.29 \cdot 48}{4} = 15.48$$

$$\Leftrightarrow n = 240.$$

Example 5:

A population has a standard deviation of 16. What is the probability that a sample mean will be within ± 2 of the population mean for each of the following sample size?

(a) $n = 64.$

(b) $n = 400.$

[Solution:]

(a)

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{16}{\sqrt{64}} = 2.$$

$$P(|\bar{X} - \mu| \le 2) = P\left(\left|\frac{\bar{X} - \mu}{\sigma_{\bar{X}}}\right| = \left|\frac{\bar{X} - \mu}{2}\right| \le \frac{2}{2}\right) \approx P(|Z| \le 1) = 0.6826.$$

(b)

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{16}{\sqrt{400}} = 0.8.$$

$$P(|\bar{X} - \mu| \le 2) = P\left(\left|\frac{\bar{X} - \mu}{\sigma_{\bar{X}}}\right| = \left|\frac{\bar{X} - \mu}{0.8}\right| \le \frac{2}{0.8}\right) \approx P(|Z| \le 2.5) = 0.9876.$$

Example 6:

In a university, 10% of the students live in the dormitories. A random sample of 100 students is selected for a particular study.

(a) What is the probability that the sample proportion (the proportion living in the dormitories) is between 0.172 and 0.178?

(b) What is the probability that the sample proportion (the proportion living in the dormitories) is greater than or equal to 0.025?

[Solution:]

$$p = 0.1, n = 100, \sigma_{\bar{P}} = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.1(1-0.1)}{100}} = 0.03.$$

(a)

$$\begin{aligned} P(0.172 \le \bar{P} \le 0.178) &= P\left(\frac{0.172 - 0.1}{0.03} \le \frac{\bar{P} - p}{\sigma_{\bar{P}}} = \frac{\bar{P} - 0.1}{0.03} \le \frac{0.178 - 0.1}{0.03}\right) \\ &\approx P(2.4 \le Z \le 2.6) \\ &= 0.0035 \end{aligned}$$

(b)

$$\begin{aligned} P(\bar{P} \ge 0.025) &= P\left(\frac{\bar{P} - p}{\sigma_{\bar{P}}} = \frac{\bar{P} - 0.1}{0.03} \ge \frac{0.025 - 0.1}{0.03}\right) \approx P(Z \ge -2.5) \\ &= 0.9938. \end{aligned}$$