Review 3

Chapter 5, 6

- Random variables and probability distribution function.
- Expected value and variance of a random variable.
- Binomial distribution Function, Poisson distribution function, and hypergeometric Distribution Function
- Normal density function and exponential density function

Equations:

$$Binomial: f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \cdots, n.$$

$$Poisson: f_X(x) = \frac{e^{-\mu}\mu^x}{x!}, x = 0, 1, 2, \cdots$$

$$Hypergeometric: f_X(x) = P(X = x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}, x = 0, 1, \cdots, r.$$

$$X \sim N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} = Z \sim N(0, 1)$$

$$Exponential: f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x}{\lambda}}, x \geq 0 \\ 0, otherwise. \end{cases}$$

Example 1:

Let X be a discrete random variable representing the hours a college student spending on reading per week. The following probability distribution has been proposed:

$$f_X(x) = \frac{x^2}{55c}, x = 1, 2, 3, 4, 5,$$

and $f_X(x) = 0$, otherwise, where c is some constant.

(a) Compute c.

(b) Compute $P(X \leq 2.2)$ and $P(X > 3.7)$.

(c) Compute $E(X)$.

(d) Compute $Var(X)$.

[Solution:]

(a)

$$\sum_{x=1}^{5} f_X(x) = \frac{1}{55c} + \frac{4}{55c} + \frac{9}{55c} + \frac{16}{55c} + \frac{25}{55c} = \frac{1}{c} = 1 \Leftrightarrow c = 1.$$

(b)

$$\begin{aligned} P(X \le 2.2) &= P(X = 1 \ \text{ or } \ X = 2) = P(X = 1) + P(X = 2) \\ &= f_X(1) + f_X(2) = \frac{1}{55} + \frac{4}{55} \\ &= \frac{1}{11} = 0.09 \end{aligned}$$

and

$$\begin{aligned} P(X > 3.7) &= P(X = 4 \ \text{ or } \ X = 5) = P(X = 4) + P(X = 5) \\ &= f_X(4) + f_X(5) = \frac{16}{55} + \frac{25}{55} \\ &= \frac{41}{55} = 0.745. \end{aligned}$$

(c)

$$\mu = E(X) = \sum_{x=1}^{5} x f_X(x) = 1 \cdot \frac{1}{55} + 2 \cdot \frac{4}{55} + 3 \cdot \frac{9}{55} + 4 \cdot \frac{16}{55} + 5 \cdot \frac{25}{55} = \frac{45}{11} = 4.09$$

(d)

$$\begin{aligned} Var(X) &= E(X^2) - \mu^2 = \textstyle\sum_{x=1}^{5} x^2 f_X(x) - \mu^2 \\ &= \left[1^2 \cdot \frac{1}{55} + 2^2 \cdot \frac{4}{55} + 3^2 \cdot \frac{9}{55} + 4^2 \cdot \frac{16}{55} + 5^2 \cdot \frac{25}{55}\right] - \left(\frac{45}{11}\right)^2 \\ &= 1.064 \end{aligned}$$

Example 2:

Twenty percent of the applications received for a particular position are rejected. What is the probability that among the next fourteen applications,

(a) none will be rejected?

(b) all will be rejected?

(c) less than 2 will be rejected?

(d) more than one will be rejected?

(e) Determine the expected number of rejected applications and its variance.

[Solution:]

Let X represent the number of rejections among the next fourteen applications. Then, the distribution function of X is

$$f_X(x) = \binom{14}{x}(0.2)^x(0.8)^{14-x}, x = 0, 1, \cdots, 14.$$

(a)

$$P(X = 0) = f_X(0) = \binom{14}{0}(0.2)^0(0.8)^{14} = (0.8)^{14} = 0.044.$$

(b)

$$P(X = 14) = f_X(14) = \binom{14}{14}(0.2)^{14}(0.8)^0 = (0.2)^{14} \cong 0.$$

(c)

$$P(X < 2) = P(X = 0) + P(X = 1) = f_X(0) + f_X(1)$$

$$= 0.044 + \binom{14}{1}(0.2)^1(0.8)^{13} = 0.044 + 14 \cdot 0.2 \cdot 0.055$$

$$\cong 0.1979.$$

(d)

$$P(X > 1) = 1 - P(X \le 1) = 1 - P(X < 2) \cong 1 - 0.1979 = 0.8021 \ \left(by \ \ (c)\right).$$

(e)

$$E(X) = np = 14 \cdot 0.2 = 2.8, Var(X) = np(1-p) = 14 \cdot 0.2 \cdot 0.8 = 2.24.$$

Example 3:

Airline passengers arrive randomly and independently at the passenger screening facility at a major international airport. The mean arrival rate is 10 passengers per minute.

(a) What is the probability of no arrivals in a 1-minute period?

(b) What is the probability that 2 or fewer passengers arrive in a 1-minute period?

(c) What is the probability of no arrivals in a 15-second period?

(d) What is the probability of at least one arrival in a 15-second period?

[Solution:]

(a) Let X represent the number of arrivals in 1 minutes. Then,

$$f_X(x) = \frac{e^{-10}10^x}{x!}, x = 0, 1, 2, \cdots$$

and $E(X) = 10$. Then,

$$P(X = 0) = f_X(0) = e^{-10}.$$

(b)

$$P(X \le 2) = f_X(0) + f_X(1) + f_X(2) = e^{-10} + 10e^{-10} + 50e^{-10} = 61e^{-10}.$$

(c) Let Y represent the number of arrivals in 15 seconds. Then,

$$f_Y(x) = \frac{e^{-2.5}2.5^x}{x!}, x = 0, 1, 2, \cdots$$

and $E(Y) = 10 \cdot \left({}^{1}/_{4}\right) = 2.5$. Then,

$$P(Y = 0) = f_Y(0) = e^{-2.5}.$$

(d)

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - f_Y(0) = 1 - e^{-2.5}.$$

Example 4:

Given that Z is a standard normal random variable and X is a normal random variable with mean 2 and standard deviation 2.

(a) $P(Z \geq 1.96)$.

(b) $P(-1.1 \leq Z \leq 1.3)$.

(c) $P\big((2c+1) \leq Z \leq 1.1\big) = 0.6881$. Find c.

(d) $P(X \geq -2)$.

(e) $P(2 \leq X \leq c) = 0.4901$. Find c.

[Solution:]

(a)

$$P(Z \geq 1.96) = P(Z \geq 0) - P(0 \leq Z < 1.96) = 0.5 - 0.475 = 0.025.$$

(b)

$$\begin{aligned} &P(-1.1 \leq Z \leq 1.3) = P(-1.1 \leq Z \leq 0) + P(0 < Z \leq 1.3) \\ &= P(0 \leq Z \leq 1.1) + P(0 < Z \leq 1.3) \ \ (symmetry\ \ of\ \ Z) \\ &= 0.3643 + 0.4032 = 0.7675. \end{aligned}$$

(c)

$$\begin{aligned} P\big((2c+1) \leq Z \leq 1.1\big) &= P\big((2c+1) \leq Z \leq 0\big) + P(0 < Z \leq 1.1) \\ &= P\big((2c+1) \leq Z \leq 0\big) + 0.3643 = 0.6881 \\ &\Leftrightarrow P\big((2c+1) \leq Z \leq 0\big) = P\big(0 \leq Z \leq -(2c+1)\big) = 0.3238 \\ &\Leftrightarrow -(2c+1) = 0.93 \\ &\Leftrightarrow \ \ c = -0.965 \end{aligned}$$

(d)

$$\begin{aligned} P(X \geq -2) &= P\left(\frac{X-2}{2} \geq \frac{-2-2}{2}\right) = P(Z \geq -2) = 0.5 + P(0 < Z \leq 2) \\ &= 0.9772 \end{aligned}$$

(e)

$$P(2 \leq X \leq c) = P\left(\frac{2-2}{2} \leq \frac{X-2}{2} \leq \frac{c-2}{2}\right) = P\left(0 \leq Z \leq \frac{c-2}{2}\right) = 0.4901$$

$$\Leftrightarrow \frac{c-2}{2} = 2.33$$

$$\Leftrightarrow c = 6.66.$$