SUPPLEMENT: UNBOUNDED NONLINEAR OPERATORS

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Abstract. One class of possibly nonlinear operators which includes bounded linear operators on a complex Hilbert space is defined. Spectral theorems for certain possibly unbounded operators can be proved based on the result for the class of operators.

1. Introduction

Operator theory has been at the heart of research in analysis (see [1]; [10], Chapter 4). Moreover, as implied by [9], considering nonlinear case should be essential. Developing useful results for the operators holds the promise for the wide applications of nonlinear functional analysis to a variety of scientific areas.

The commonly used operators might not be bounded. For examples, the linear multiplication operator and the linear differentiation operator (see [6], Chapter 10.7) which are related to the position operator and the momentum operator (see [6], Chapter 11) in quantum mechanics are not bounded.

Spectral theory is one of the main topics of modern functional analysis and its applications (see [6]; [14]). Spectral theory for certain classes of linear operators, including compact, symmetric, unitary, or normal operators, has been well developed (see [5]; [7]), particularly in a Hilbert space. Spectral theory for the nonlinear operators is an emerging field in functional analysis (see [3]). In [12], the spectral theorems for certain possibly nonlinear operators which are referred to as the generalized real definite operators and include bounded linear symmetric operators as special cases can be proved. In this article, a class of operators which is referred to as the generalized complex definite operators and includes the class of the bounded generalized real definite operators is defined and the main result given in [12] is extended to the associated bounded generalized complex definite operators in next section. Based on the spectral theorem for the generalized complex definite operators, the spectral theorems of the possibly unbounded nonlinear operators of interest can be proved in Section 3.

From now on $D(F)$ and $R(F)$ are denoted as the domain and the range of an operator F, respectively, and the notation $||\cdot||_Z$ is denoted as the norm of the normed space Z. The space of interest is the normed space implicitly.

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On the other hand, the other function spaces will be indicated explicitly. Note that vector spaces and normed spaces of interest in this article are not trivial, i.e., not only including the zero element. Also, let $S_2 \backslash S_1$ denote the intersection of the set S_2 and the complement of the set S_1 and let the notation of the composition of two operators be $\circ$.

2. Generalized complex definite operators

Since the main results in this article are based on the quasi-product, the generalized real definite operators, the generalized eigenvector, and the projection operator in [12], their definitions and properties along with the normed spaces of interest are summarized as follows.

- Let $V(S, Y)$ be the set of all operators from the set $S \subset X$ into Y, $0 \in S$, i.e., the set of arbitrary maps from S into Y, where X and Y are normed spaces over a field K with some sensible norms and where K is either the real field R or the complex field C. Also let the zero element in $V(S, Y)$ be the operator of which image equal to the zero element in Y. An application of algebraic operations to elements $F_1, F_2 \in V(S, Y)$ gives the operators $F_1 + F_2$ and αF_1 from S into Y with $(F_1 + F_2)(x) = F_1(x) + F_2(x)$ and $(\alpha F_1)(x) = \alpha F_1(x)$ for $x \in S$, where $\alpha \in K$ is a scalar. Then $V(S, Y)$ is a vector space. Let $B(S, Y)$ be the subset of $V(S, Y)$ with the property that $||F||_{B(S,Y)}$ is finite for all $F \in B(S, Y)$, where
$$||F||_{B(S,Y)} = \max\left(\sup_{x \neq 0, x \in S} \frac{\|F(x)\|_Y}{\|x\|_X}, \|F(0)\|_Y \right).$$
Note that $B(S, Y)$ is a normed space, i.e., $||\cdot||_{B(S,Y)}$ being a normed function on $B(S, Y)$. As $X = Y$, the notations $V(S) = V(S, X)$ and $B(S) = B(S, X)$ are used.
- A quasi-product $[\cdot, \cdot]_S$ on S is a mapping (or a map) of $S \times S$ into the scalar field K with the following properties:
(a)
$$[x, x]_S \geq 0$$
for $x \in S$.
(b)
$$|[x, y]_S| \leq c \, ||x||_X \, ||y||_X$$
for $x, y \in S$, where c is a positive number.
(c)
$$\left[\sum_{i=1}^{n} \alpha_i x_i, y \right]_S = c(y) \sum_{i=1}^{n} \alpha_i \, [x_i, y]_S$$

for any $n \geq 1$, $x_1, \ldots, x_n, y, \sum_{i=1}^{n} \alpha_i x_i \in S$ and $\alpha_1, \ldots, \alpha_n \in K$, where $c : S \to R$ is a positive bounded function and is bounded away from 0.

- An operator $F : D(F) \to X$ is generalized real definite if and only if there exist a quasi-product $[\cdot, \cdot]_X$ and an operator $g : D(F) \to X$ satisfying $g(x) \neq 0$ for $x \neq 0$ and
$$[F(x), g(x)]_X \in R$$
for $x \in D(F)$, where $D(F) \subset X$. Furthermore, F is g-positive, denoted by $F \geq 0$, if and only if
$$[F(x), g(x)]_X \geq 0$$
for $x \in D(F)$. For the operators $F_1 : D(F_1) \to X$ and $F_2 : D(F_1) \to X$,
$$F_1 \geq F_2$$
if and only if
$$F_1 - F_2 \geq 0,$$
where $D(F_1) \subset X$. The operator $|F|$ is defined by
$$|F|(x) = F(x)$$
if $[F(x), g(x)]_X \geq 0$ and
$$|F|(x) = -F(x)$$
if $[F(x), g(x)]_X < 0$ for $x \in D(F)$. The positive part of F is
$$F^+ = \frac{|F| + F}{2}$$
and the negative part of F is
$$F^- = \frac{|F| - F}{2}.$$
If F is a symmetric linear operator on a Hilbert space, the operator g is the identity map, and the quasi-product is the inner product on the Hilbert space, F is generalized real definite and F being positive implies F being g-positive. Therefore, the generalized real definiteness and the g-positivity extend the notions of the symmetry and the positivity respectively. Note that a generalized real definite operator F might not be bounded, i.e., not lying in $B[D(F)]$.
- Let $F : D(F) \to X$ be an operator and $\gamma : D(F) \to X$ be a g-positive operator satisfying $[\gamma(x), g(x)]_X = k_1(x)||x||_X||g(x)||_X$ and $||\gamma(x)||_X = k_2(x)||x||_X$ for $x \in D(F)$, where $D(F) \subset X$, both k_1 and k_2 are positive bounded functions and are bounded away from 0. The g-resolvent set of F, denoted by $\rho(F)$ and $\rho(F) \subset C$, consists of the scalars λ such that $R_\lambda = (F - \lambda\gamma)^{-1}$ exists (see [6], A1.2), is bounded, and $D(R_\lambda)$ is a dense set of X. The set $\sigma(F) = C \backslash \rho(F)$ is referred to as the g-spectrum of F. As $F(x) = \lambda\gamma(x)$ for some

$x \neq 0$, x is referred to as the g-eigenvector of F corresponding to the g-eigenvalue λ.

- The projection operator $E_S \in B[D(E_S)]$ corresponding to the set $S \subset D(E_S)$ is defined by $E_S(x) = x$ if $x \in S$ and $E_S(x) = 0$ otherwise, where $D(E_S) \subset X$.

Any bounded linear operator T on a complex Hilbert space can be decomposed as $T = T_r + iT_c$, where both T_r and T_c are linear symmetric operators. In addition, the spectral resolution of a normal linear operator is the integral over some bounded region of the complex plane (see [11], Section 111; [13], Chapter XI). In this section, the nonlinear generalization of the above bounded linear operator is given and is referred to as the generalized complex definite operator. Furthermore, the spectral resolution of certain generalized complex definite operators has the form similar to the one of normal linear operators as stated in Theorem 2.5.

Remark 2.1. For a Banach algebra over the complex field with an involution (see [2], Chapter 6), any element a in the Banach algebra can be decomposed as $a = a_r + ia_c$, where both a_r and a_c are self-adjoint.

It is natural to define the generalized complex definite operator as follows.

Definition 2.2. Let X be a complex normed space. An operator $F : D(F) \to X$ is generalized complex definite if and only if

$$F = F_r + iF_c,$$

where $D(F) \subset X$, both F_r and F_c are generalized real definite with respect to the same operator g on $D(F)$ and the same quasi-product $[\cdot,\cdot]_X$ on X. F_r is referred to as the real part of F, while F_c is referred to as the imaginary part of F.

From now on let $N(F)$ be the null space (set) of an operator F, i.e., $N(F) = \{x : F(x) = 0, x \in D(F)\}$. Further, let $F_{rs_{ri}} = F_r - s_{ri}\gamma$, and $F_{cs_{cj}} = F_c - s_{cj}\gamma$, where $s_{ri}, s_{cj} \in R$. Let $E_{s_{ri}}$ and $E_{s_{cj}}$ be the projection operators corresponding to $N(F^+_{rs_{ri}})$ and $N(F^+_{cs_{cj}})$, respectively. In addition, let $\Delta = \mu - \lambda$ and $E_\Delta = E_\mu - E_\lambda$, where $\lambda, \mu \in R$ and $\lambda < \mu$.

The following lemma, a counterpart of Lemma 3.19 in [12], gives the existence of the spectral integral. Since the proof of the lemma is similar to its counterpart, the proof is not presented.

Lemma 2.3. *Let $F \in V(S)$ be generalized complex definite, $F(0) = 0$, and $F_r, F_c \in B(S)$, where X is a complex Banach space. There exist a bounded interval $[m_r, M_r]$ with any partition $\{s_{ri}\}$ satisfying $m_r = s_{r0} < s_{r1} < \cdots < s_{rn} = M_r$, $\Delta_{ri} = s_{ri} - s_{r(i-1)} < \epsilon_{nm}$ and a bounded interval $[m_c, M_c]$ with any partition $\{s_{cj}\}$ satisfying $m_c = s_{c0} < s_{c1} < \cdots < s_{cm} = M_c$, $\Delta_{cj} = s_{cj} - s_{c(j-1)} < \epsilon_{nm}$ such that $F_{nm} = \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}(\gamma \circ E_{\Delta_{ij}})$ converges to a nonlinear operator in $B(S)$ with respect to the norm topology $||\cdot||_{B(S)}$*

and the convergence is independent of the choice of $z_{ij} = \lambda_{ri} + i\lambda_{cj}$ *with* $\lambda_{ri} \in (s_{r(i-1)}, s_{ri}]$ *and* $\lambda_{cj} \in (s_{c(j-1)}, s_{cj}]$ *as* $n, m \to \infty$, *where* $E_{m_r}(x) = 0$, $E_{m_c}(x) = 0$ *for* $x \in S$, $E_{M_r} = I$, $E_{M_c} = I$, $E_{\Delta_{ij}} = E_{\Delta_{ri}} \circ E_{\Delta_{cj}}$, *and* $0 < \epsilon_{nm} \underset{n,m\to\infty}{\longrightarrow} 0$, *and where* I *is the identity map on* S.

Based on the above lemma, the associated spectral integral can be defined thus.

Definition 2.4. Let X be a complex Banach space, $F \in V(S)$ be generalized complex definite, $F(0) = 0$, $F_r, F_c \in B(S)$, the partition $\{s_{ri}\}$ satisfy $m_r = s_{r0} < s_{r1} < \cdots < s_{rn} = M_r$, $\Delta_{ri} = s_{ri} - s_{r(i-1)} < \epsilon_{nm}$, and the partition $\{s_{cj}\}$ satisfy $m_c = s_{c0} < s_{c1} < \cdots < s_{cm} = M_c$, $\Delta_{cj} = s_{cj} - s_{c(j-1)} < \epsilon_{nm}$, where $0 < \epsilon_{nm} \underset{n,m\to\infty}{\longrightarrow} 0$. If $\sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}(\gamma \circ E_{\Delta_{ij}})$ converges to an operator in the sense of operator convergence, i.e., with respect to the norm topology $||\cdot||_{B(S)}$, and the convergence is independent of the choice of $z_{ij} = \lambda_{ri} + i\lambda_{cj}$, $\lambda_{ri} \in (s_{r(i-1)}, s_{ri}]$ and $\lambda_{cj} \in (s_{c(j-1)}, s_{cj}]$ as $n, m \to \infty$, the limit operator is denoted as $\int_\Omega zd(\gamma \circ E_z)$, where $E_{\Delta_{ij}} = E_{\Delta_{ri}} \circ E_{\Delta_{cj}}$ and $\Omega = \{z = \lambda_r + i\lambda_c : (\lambda_r, \lambda_c) \in [m_r, M_r] \times [m_c, M_c]\}$ is a bounded rectangle of the complex plane.

The following inequalities, based on Lemma 3.18 (c) in [12], give the relations between the generalized real definite operators F_r and F_c and the function γ, i.e., the approximated decomposition of F_r and F_c in terms of γ,

$$s_{r(i-1)}(\gamma \circ E_{\Delta_{ri}}) \le F_r \circ E_{\Delta_{ri}} \le s_{ri}(\gamma \circ E_{\Delta_{ri}})$$

and

$$s_{c(j-1)}(\gamma \circ E_{\Delta_{cj}}) \le F_c \circ E_{\Delta_{cj}} \le s_{cj}(\gamma \circ E_{\Delta_{cj}}).$$

Using the above inequalities gives the following theorem.

Theorem 2.5. *Let* X *be a complex Banach space,* $F \in V(S)$ *be generalized complex definite,* $F(0) = 0$, *and* $F_r, F_c \in B(S)$. *Then*

$$[F(x), g(x)]_X = \left[\left[\int_\Omega zd(\gamma \circ E_z)\right](x), g(x)\right]_X$$

for $x \in S$, *where* Ω *depending on* F *is a bounded rectangle of the complex plane.*

Proof. $\int_\Omega zd(\gamma \circ E_z)$ exists by Lemma 2.3. Because $F = \sum_{i=1}^{n}\sum_{j=1}^{m} F \circ E_{\Delta_{ij}}$, hence

$$\sum_{i=1}^{n}\sum_{j=1}^{m} s_{ri}(\gamma \circ E_{\Delta_{ij}}) - F_r \le \sum_{i=1}^{n}\sum_{j=1}^{m} \Delta_{ri}(\gamma \circ E_{\Delta_{ij}}) \le \epsilon_{nm} \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma \circ E_{\Delta_{ij}}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{m} s_{cj}(\gamma \circ E_{\Delta_{ij}}) - F_c \le \sum_{i=1}^{n}\sum_{j=1}^{m} \Delta_{cj}(\gamma \circ E_{\Delta_{ij}}) \le \epsilon_{nm} \sum_{i=1}^{n}\sum_{j=1}^{m} \gamma \circ E_{\Delta_{ij}}.$$

Thus, there exist positive numbers k_1 and k_2 such that for $\lambda_{ri} = s_{ri}$ and $\lambda_{cj} = s_{cj}$,

$$\begin{aligned}
& \left| \left[\sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} (\gamma \circ E_{\Delta_{ij}})(x) - F(x), g(x) \right]_X \right| \\
= & \left| \left[\left(\sum_{i=1}^{n} \sum_{j=1}^{m} s_{ri} (\gamma \circ E_{\Delta_{ij}})(x) - F_r(x) \right) \right. \right. \\
& \left. \left. + i \left(\sum_{i=1}^{n} \sum_{j=1}^{m} s_{cj} (\gamma \circ E_{\Delta_{ij}})(x) - F_c(x) \right), g(x) \right]_X \right| \\
\le & \; k_1 \left\{ \left| \left[\sum_{i=1}^{n} \sum_{j=1}^{m} s_{ri} (\gamma \circ E_{\Delta_{ij}})(x) - F_r(x), g(x) \right]_X \right| \right. \\
& \left. + \left| \left[\sum_{i=1}^{n} \sum_{j=1}^{m} s_{cj} (\gamma \circ E_{\Delta_{ij}})(x) - F_c(x), g(x) \right]_X \right| \right\} \\
\le & \; 2k_1 \left[\epsilon_{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} (\gamma \circ E_{\Delta_{ij}})(x), g(x) \right]_X \\
= & \; 2k_1 \left[\epsilon_{nm} \gamma(x), g(x) \right]_X \\
\le & \; 2k_1 k_2 \epsilon_{nm} \left|\left| x \right|\right|_X \left|\left| g(x) \right|\right|_X
\end{aligned}$$

for $x \in S$. $[(\int_\Omega zd(\gamma \circ E_z) - F)(x), g(x)]_X = 0$ by the continuity of the quasi-product and hence

$$\left[\int_\Omega zd(\gamma \circ E_z)(x), g(x) \right]_X = \left[F(x), g(x) \right]_X .$$

$\square$

Note that

$$F = \int_\Omega zd(\gamma \circ E_z)$$

also holds by imposing some conditions on the quasi-products (see [12]).

3. Nonlinear spectral analysis

In Section 3.1, two examples of the unbounded nonlinear operators associated with the linear multiplication and the linear differentiation operators (see [6], Chapter 10.7) which are related to the position operator and the momentum operator (see [6], Chapter 11), respectively, in quantum mechanics are given. Furthermore, spectral analysis for the possibly unbounded nonlinear operators is given in Section 3.2.

3.1. **Examples.** Let $L_p(-\infty,\infty), 1 \le p < \infty$, be the spaces of all complex-valued functions x defined on $(-\infty,\infty)$ satisfying that $|x|^p$ is integrable with respect to the Lebesgue measure. In addition, for $x_1, x_2 \in L_1(-\infty,\infty)$ the quasi-product is defined by

$$[x_1, x_2]_{L_1(-\infty,\infty)} = \int_{-\infty}^{\infty} x_1(t)dt \overline{\int_{-\infty}^{\infty} x_2(t)dt},$$

where $\overline{z}$ is the conjugate of the complex number z.

Example 3.1. Let the linear operator $T_m : S_{T_m} \to L_2(-\infty,\infty)$ defined by $T_m(x) = x_0 x$ for $x \in S_{T_m}$, where $x_0(t) = t$ for $t \in (-\infty,\infty)$ and S_{T_m}, a subset of $L_2(-\infty,\infty)$, consists of all functions satisfying $x_0 x \in L_2(-\infty,\infty)$. The relevant nonlinear operator $F_m : S_{F_m} \to L_1(-\infty,\infty)$ defined by $F_m(x) = x_0|x|^2$ for $x \in S_{F_m}$, where S_{F_m} consisting of all functions satisfying $x_0|x|^2 \in L_1(-\infty,\infty)$ is a subset of $L_2(-\infty,\infty)$. Note that $S_{T_m} \subset S_{F_m}$. Both T_m and F_m are unbounded because for x_n defined by $x_n(t) = 1$ for $(n-1) \le t \le n$ and $x_n(t) = 0$ elsewhere,

$$\frac{\|T_m(x_n)\|_{L_2(-\infty,\infty)}}{\|x_n\|_{L_2(-\infty,\infty)}} = [(n-1)n + 1/3]^{1/2}$$

and

$$\frac{\|F_m(x_n)\|_{L_1(-\infty,\infty)}}{\|x_n\|_{L_2(-\infty,\infty)}} = n - 1/2$$

for every positive n. Note that

$$< T_m(e), e >_{L_2(-\infty,\infty)} = \left[F_m(e), |e|^2\right]_{L_1(-\infty,\infty)}$$

for any unit vector $e \in S_{T_m}$, where $< \cdot, \cdot >_{L_2(-\infty,\infty)}$ is the inner product on $L_2(-\infty,\infty)$. $e \in S_{T_m}$ is referred to as the state function (or the wave function) and T_m is the operator corresponding to the observable (see [4], Chapter 2; [6], Chapter 11.1; [8], Chapter 7) in quantum mechanics.

Example 3.2. Let the linear operator $T_d : S_{T_d} \to L_2(-\infty,\infty)$ defined by $T_d(x) = ix'$ for $x \in S_{T_d}$, where x' is the derivative of x and S_{T_d}, a subset of $L_2(-\infty,\infty)$, consists of all functions satisfying $x' \in L_2(-\infty,\infty)$. The relevant nonlinear operator $F_d : S_{F_d} \to L_1(-\infty,\infty)$ defined by $F_d(x) = ix'\bar{x}$ for $x \in S_{F_d}$, where S_{F_d}, a subset of $L_2(-\infty,\infty)$, consists of all functions satisfying $x'\bar{x} \in L_1(-\infty,\infty)$, and where $\bar{x}$ is the conjugate function of x. Note that $S_{T_d} \subset S_{F_d}$. Both T_d and F_d are unbounded because for x_n defined by $x_n(t) = nt$ for $0 \le t \le 1/n$ and $x_n(t) = 0$ elsewhere,

$$\frac{\|T_d(x_n)\|_{L_2(-\infty,\infty)}}{\|x_n\|_{L_2(-\infty,\infty)}} = \sqrt{3}n$$

and

$$\frac{\|F_d(x_n)\|_{L_1(-\infty,\infty)}}{\|x_n\|_{L_2(-\infty,\infty)}} = \frac{\sqrt{3n}}{2}$$

for every positive n. Similar to the previous example,

$$< T_d(e), e >_{L_2(-\infty,\infty)} = \left[F_d(e), |e|^2\right]_{L_1(-\infty,\infty)}$$

for any unit vector $e \in S_{T_d}$, where e is the wave function and T_d is associated with the observable (see [6], Chapter 11.1) in quantum mechanics.

3.2. **Spectral analysis.** In this subsection, both the spectral resolutions of the possibly unbounded nonlinear operators in terms of the quasi-products and a result related to the g-resolvent set are given.

3.2.1. *Nonlinear Cayley transform.* The spectral theorem of the self-adjoint linear operator on a Hilbert space relies on the Cayley transform (see [6], Chapter 10.6). Analogously, the nonlinear Cayley transform plays a crucial role in the spectral theorem of the nonlinear operators. The boundedness of the nonlinear Cayley transform is stated in Lemma 3.5, while the equation for the operator and its Cayley transform is given in Theorem 3.6. The lemma and the theorem can be used to give the spectral resolution of the possibly unbounded nonlinear operator in terms of the quasi-products.

Definition 3.3. Let X be a complex normed space. The operational Cayley transform of $F \in V(S)$ is $G_\zeta : R(F_{-i}) \to X$ defined by $G_\zeta = F_i \circ F_{-i}^{-1}$, where $F_{-i} : S \to X$ defined by $F_{-i} = F + i\zeta$ is injective, $\zeta \in V(S)$ with $\zeta(0) = 0$, and $F_i : S \to X$ is defined by $F_i = F - i\zeta$.

Note that the existence of the operational Cayley transform depends on the injectiveness of F_{-i}. An operator can be injective as the operator is bounded below defined as follows.

Definition 3.4. An operator $F \in V(S,Y)$ is bounded below if and only if there exists a positive number $\underline{k}$ such that

$$||F(x_1) - F(x_2)||_Y \geq \underline{k}\, ||x_1 - x_2||_X$$

for $x_1, x_2 \in S$.

The main results of this subsection are given below.

Lemma 3.5. *Let $F \in V(S)$ and $F(0) = 0$. If $F_{-i} = F + i\zeta$ is bounded below, then the operational Cayley transform G_ζ of F is bounded.*

Proof. The injectiveness of F_{-i} is proved first. Suppose not, then there exist $x, y \in X, x \neq y$, such that $F_{-i}(x) = F_{-i}(y)$. Hence,

$$0 = ||F_{-i}(x) - F_{-i}(y)||_X \geq \underline{k}\, ||x - y||_X > 0,$$

a contradiction, where $\underline{k}$ is some positive number. Therefore, F_{-i}^{-1} exists. Let $x^* = F_{-i}^{-1}(x)$ for $x \in R(F_{-i})$. Because F_{-i} is bounded below,

$$\begin{aligned} & ||G_\zeta(x)||_X \\ = \ & ||F_i(x^*)||_X \\ \le \ & ||(F_i - F_{-i})(x^*)||_X + ||F_{-i}(x^*)||_X \\ \le \ & \overline{k}\, ||F_{-i}(x^*)||_X \\ = \ & \overline{k}\, ||x||_X , \end{aligned}$$

i.e., $||G_\zeta||_{B[R(F_{-i}),X]} \le \overline{k}$, where $\overline{k}$ is some positive number.

□

Theorem 3.6. *Let G_ζ be the operational Cayley transform of the operator $F \in V(S)$. If both $F_{-i} = F + i\zeta$ and ζ are injective, then*

$$F = \left(\frac{I + G_\zeta}{2}\right) \circ \left[(I - G_\zeta)^{-1}\right] \circ (2i\zeta),$$

where $I : R(F_{-i}) \to X$ is the identity map.

Proof. Since $(G_\zeta \circ F_{-i})(x) = F(x) - i\zeta(x) = F_i(x)$ for $x \in S$, hence

$$[(I + G_\zeta) \circ F_{-i}](x) = F_{-i}(x) + (G_\zeta \circ F_{-i})(x) = 2F(x)$$

and

$$[(I - G_\zeta) \circ F_{-i}](x) = F_{-i}(x) - (G_\zeta \circ F_{-i})(x) = 2i\zeta(x).$$

If $(I - G_\zeta)^{-1}$ exists, then

$$F_{-i}(x) = \left[(I - G_\zeta)^{-1} \circ (2i\zeta)\right](x)$$

and thus

$$F = \left(\frac{I + G_\zeta}{2}\right) \circ \left[(I - G_\zeta)^{-1} \circ (2i\zeta)\right].$$

It remains to prove the existence of $(I - G_\zeta)^{-1}$, i.e., $I - G_\zeta$ being injective. For $y_1, y_2 \in D(G_\zeta)$, there exist $x_1, x_2 \in S$ such that $F_{-i}(x_1) = y_1$ and $F_{-i}(x_2) = y_2$. Then if

$$(I - G_\zeta)(y_1) = (I - G_\zeta)(y_2) = 2i\zeta(x_1) = 2i\zeta(x_2),$$

then $F_{-i}(x_1) = y_1 = y_2 = F_{-i}(x_2)$ owing to ζ being injective and $x_1 = x_2$ thus. Therefore, $I - G_\zeta$ is injective and $(I - G_\zeta)^{-1}$ exists.

□

3.2.2. *Spectral analysis.* In the following, Lemma 3.7 gives the result related to the transforms of the projection operators. Lemma 3.7 can be used to prove Lemma 3.8, a "transformed" expression of a nonlinear operator. Based on Lemma 3.8, Theorem 3.9 gives the spectral representation in terms of the quasi-products for the possibly unbounded nonlinear operators. Let $H(S_1) = \{H(x) : x \in S_1\}$, where $H \in V(S)$ and $S_1 \subset S$.

Lemma 3.7. *Let $H \in V(S)$, $H(0) = 0$, and H^{-1} exists. Then*

$$E_{S_1}(x) = \left(H^{-1} \circ E_{H(S_1)}\right)[H(x)],$$

i.e.,

$$E_{H(S_1)}[H(x)] = (H \circ E_{S_1})(x),$$

where S_1 containing 0 is a subset of S, the projection operator E_{S_1} is defined on S, and the projection operator $E_{H(S_1)}$ is defined on $H(S)$.

Proof. As $x \in S_1$, then $H(x) \in H(S_1)$ and thus

$$E_{S_1}(x) = x = \left(H^{-1} \circ H\right)(x) = \left(H^{-1} \circ E_{H(S_1)}\right)[H(x)].$$

As $x \notin S_1$, then $H(x) \notin H(S_1)$ since otherwise there exists a $x^* \in S_1$ such that $H(x^*) = H(x)$, a contradiction to the injectiveness of H. Therefore,

$$E_{S_1}(x) = 0 = H^{-1}(0) = \left(H^{-1} \circ E_{H(S_1)}\right)[H(x)].$$

□

Lemma 3.8. *Let X be a complex Banach space and $F \in V(S)$ be generalized complex definite with $F(0) = 0$, $H \in V(S)$ with $H(0) = 0$, and H^{-1} exist. Then*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}\left(\gamma \circ E_{\Delta_{ij}}\right)(x) = \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}\left(\gamma_{H^{-1}} \circ E_{H(\Delta_{ij})}\right)(y)$$

for $x \in S, z_{ij} \in C$, where $E_{\Delta_{ij}}$ are the projection operators corresponding to F given in Definition 2.4, $E_{H(\Delta_{ij})}$ are the projection operators defined on $H(S)$ corresponding to the sets $H[R(E_{\Delta_{ij}})]$, $\gamma_{H^{-1}} = \gamma \circ H^{-1}$, and $y = H(x)$.

Proof. It suffices to prove that $(\gamma \circ E_{\Delta_{ij}})(x) = (\gamma_{H^{-1}} \circ E_{H(\Delta_{ij})})(y)$. By Lemma 3.7,

$$\begin{aligned} & \left(\gamma_{H^{-1}} \circ E_{H(\Delta_{ij})}\right)(y) \\ = & \left[\gamma \circ \left(H^{-1} \circ E_{H(\Delta_{ij})}\right)\right](y) \\ = & \left(\gamma \circ E_{\Delta_{ij}}\right)(x). \end{aligned}$$

□

Theorem 3.9. *Let X be a complex Banach space, $F \in V(S)$, and $F(0) = 0$. If the operational Cayley transform $G_\zeta = G_{\zeta r} + iG_{\zeta c}$ of F is generalized complex definite with the generalized real definite operators $G_{\zeta r}, G_{\zeta c} \in B[R(F_{-i}), X]$ and both $F_{-i} = F + i\zeta$ and ζ are injective, then*

$$[F(x), k(x)]_X = \left[\left[\int_\Omega zd\left(\gamma_{H^{-1}} \circ E_{H(z)}\right)\right](x), k(x)\right]_X$$

for $x \in S$, where Ω is the bounded rectangle of the complex plane depending on the operator $(I + G_\zeta)/2$, z_{ij} are the associated grid points in Ω, $H = (-i/2)[\zeta^{-1} \circ (I - G_\zeta)]$, $\gamma_{H^{-1}} = \gamma \circ H^{-1}$, $k \circ H = g$, and

$$\lim_{n,m\to\infty} \left[\sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} \left(\gamma_{H^{-1}} \circ E_{H(\Delta_{ij})} \right) \right] (x) = \left[\int_\Omega zd \left(\gamma_{H^{-1}} \circ E_{H(z)} \right) \right] (x),$$

i.e., the limit being in sense of pointwise convergence, and where $E_{H(\Delta_{ij})}$ defined on $H[R(F_{-i})]$ is the projection operator corresponding to the sets $H[R(E_{\Delta_{ij}})]$ and $E_{\Delta_{ij}}$ defined on $R(F_{-i})$ is the projection operator corresponding to the operator $(I + G_\zeta)/2$.

Proof. Let $x = H(y)$ for $y \in R(F_{-i})$. $F(x) = [(I + G_\zeta)/2](y)$ for $x \in S$ by Theorem 3.6 and then by the spectral representation for the bounded operator $(I + G_\zeta)/2$, i.e., Theorem 2.5, and by Lemma 3.8,

$$\begin{aligned}
& [F(x), k(x)]_X \\
= \quad & \left[\left[\int_\Omega zd \left(\gamma \circ E_z \right) \right] (y), g(y) \right]_X \\
= \quad & \lim_{n,m\to\infty} \left[\left[\sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} \left(\gamma \circ E_{\Delta_{ij}} \right) \right] (y), g(y) \right]_X \\
= \quad & \left[\lim_{n,m\to\infty} \left[\sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} \left(\gamma_{H^{-1}} \circ E_{H(\Delta_{ij})} \right) \right] (x), k(x) \right]_X \\
= \quad & \left[\left[\int_\Omega zd \left(\gamma_{H^{-1}} \circ E_{H(z)} \right) \right] (x), k(x) \right]_X .
\end{aligned}$$

□

3.2.3. *G-resolvent set.* The main results of this subsection, Theorem 3.12 and Corollary 3.13, are about the g-resolvent set and the spectral resolution of some nonlinear operators. The following lemma and corollary can be used to prove Theorem 3.12. The proof of the lemma is quite routine and is not presented.

Lemma 3.10. *Let the operator $F \in V(S, Y)$ be bounded below. Then F is injective.*

Corollary 3.11. *Let the operator $F \in V(S, Y)$ be bounded below and $F(0) = 0$. Then $F^{-1} \in B[R(F), X]$.*

Proof. Since F is bounded below, F is injective by Lemma 3.10 and F^{-1} exists thus. Because F is bounded below,

$$||F^{-1}||_{B[R(F),X]} = \sup_{y\neq 0, y\in R(F)} \frac{||F^{-1}(y)||_X}{||y||_Y} = \sup_{F(x)\neq 0, x\in S} \frac{||x||_X}{||F(x)||_Y} \leq \frac{1}{\underline{k}},$$

where $\underline{k}$ is some positive number.

□

Theorem 3.12. *Let X be a complex normed space, the operator $F \in V(S)$, $F(0) = 0$, $\eta \in V(S)$ be bounded below with $\eta(0) = 0$, both $\hat{F}(x_1, x_2) = F(x_1) - F(x_2)$ and $\hat{\eta}(x_1, x_2) = \eta(x_1) - \eta(x_2)$ be generalized real-definite with respect to the same quasi-product and the operator g on $S \times S$, and $\hat{\eta}$ satisfy*

$$[\hat{\eta}(x_1, x_2), g(x_2, x_2)]_X = k_1(x_1, x_2) \, ||\hat{\eta}(x_1, x_2)||_X \, ||g(x_1, x_2)||_X$$

for $x_1, x_2 \in S$, where k_1 is a positive function on $S \times S$ and is bounded away from 0. Then $F_z = F - z\eta$ is bounded below and $F_z^{-1} \in B[R(F_z), X]$ for $z \in C \backslash R$. Further, if $\eta = \gamma$ and $R(F_z)$ is a dense set of X, then $C \backslash R \subset \rho(F)$.

Proof. Let $z = a + bi, b \neq 0$. If F_z is bounded below, $F_z^{-1} \in B[R(F_z), X]$ by Corollary 3.11. Further, $C \backslash R \subset \rho(F)$ as $\eta = \gamma$ and $R(F_z)$ is a dense set of X. It remains to prove that F_z is bounded below. Because $\hat{F}$ and $\hat{\eta}$ are generalized real-definite, then for $x_1, x_2 \in S$,

$$\begin{aligned} & [F_z(x_1) - F_z(x_2), g(x_1, x_2)]_X \\ = \quad & k \left\{ \left[\hat{F}(x_1, x_2) - a\hat{\eta}(x_1, x_2), g(x_1, x_2) \right]_X - bi \, [\hat{\eta}(x_1, x_2), g(x_1, x_2)]_X \right\} \end{aligned}$$

and thus

$$Im \left\{ [F_z(x_1) - F_z(x_2), g(x_1, x_2)]_X \right\} = -kb \, [\eta(x_1) - \eta(x_2), g(x_1, x_2)]_X \, ,$$

where k greater than some positive constant is a positive number depending on the values of $g(x_1, x_2)$ and $Im(z)$ is the imaginary part of the complex number z. Then

$$|[F_z(x_1) - F_z(x_2), g(x_1, x_2)]_X| \geq \underline{k}k \, |b| \, ||x_1 - x_2||_X \, ||g(x_1, x_2)||_X$$

owing to η being bounded below and

$$|[F_z(x_1) - F_z(x_2), g(x_1, x_2)]_X| \leq \tilde{k}| \, ||F_z(x_1) - F_z(x_2)||_X \, ||g(x_1, x_2)||_X \, ,$$

where $\underline{k}$ and $\tilde{k}$ are some positive numbers. Therefore, F_z is bounded below by the two inequalities.

□

A linear self-adjoint operator defined on a subset containing 0 of a Hilbert space is a special case of the operator F in Theorem 3.12 if $g(x_1, x_2) = x_1 - x_2$. In addition, the above theorem indicates that $F_{-i} = F + i\zeta$ involved in the operational Cayley transform G_ζ is bounded below and thus the operational Cayley transform is bounded by imposing some conditions on F and ζ. Then, the spectral resolution of some possibly unbounded operators in terms of the quasi-products can be proved based on Theorem 3.9 and Theorem 3.12, as indicated by the following corollary.

Corollary 3.13. *Let X be a complex Banach space, $F \in V(S)$, $F(0) = 0$, and G_ζ be the operational Cayley transform of F. If ζ is bounded below, both $\hat{F}(x_1, x_2) = F(x_1) - F(x_2)$ and $\hat{\zeta}(x_1, x_2) = \zeta(x_1) - \zeta(x_2)$ are generalized real-definite with respect to the same quasi-product and the operator g on $S \times S$, and $\hat{\zeta}$ satisfies*

$$\left[\hat{\zeta}(x_1, x_2), g(x_2, x_2)\right]_X = k_1(x_1, x_2) \left|\left|\hat{\zeta}(x_1, x_2)\right|\right|_X ||g(x_1, x_2)||_X$$

for $x_1, x_2 \in S$, then

$$[F(x), k(x)]_X = \left[\left[\int_\Omega zd\left(\gamma_{H^{-1}} \circ E_{H(z)}\right)\right](x), k(x)\right]_X$$

for $x \in S$, where Ω is the bounded rectangle of the complex plane depending on the operator $(I + G_\zeta)/2$, $H = (-i/2)[\zeta^{-1} \circ (I - G_\zeta)]$, $\gamma_{H^{-1}} = \gamma \circ H^{-1}$, $\int_\Omega zd\left(\gamma_{H^{-1}} \circ E_{H(z)}\right)$ is the limit operator given in Theorem 3.9, k_1 is a positive function on $S \times S$ and is bounded away from 0, and $k \circ H = g_1 \circ F_{-i}^{-1}$, and where $g_1(x) = g(x, 0)$.

Proof. F_{-i} is bounded below and $F_{-i}^{-1} \in B[R(F_{-i}), X]$ by Theorem 3.12. Hence, the operational Cayley transform G_ζ is bounded by Lemma 3.5. Let $y = F_{-i}^{-1}(x)$ for $x \in R(F_{-i})$. Then $G_\zeta(x) = F_i(y) = F(y) - i\zeta(y)$. Let $G_{\zeta r}(x) = F(y)$ and $G_{\zeta c}(x) = -\zeta(y)$. Then $G_{\zeta r}$ and $G_{\zeta c}$ are generalized real-definite with respect to $g_1 \circ F_{-i}^{-1}$ and are bounded owing to G_ζ and $\zeta \circ F_{-i}^{-1}$ being bounded. By Theorem 3.9, the result holds. □

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