CHAPTER 9

THE CAPITAL ASSET PRICING MODEL
THE CAPITAL ASSET PRICING MODEL

- The Capital Asset Pricing Model
- Extensions of the APM
- The CAPM and Liquidity
The capital asset pricing model is a set of predictions concerning equilibrium expected returns on risky assets. Harry Markowitz laid down the foundation of modern portfolio management in 1952. The CAPM was developed 12 years later in articles by William Sharpe, John Lintner, and Jan Mossin.
We will approach the CAPM by posing the question “what if,” where the “if” part refers to a simplified world. Positing an admittedly unrealistic world allows a relatively easy leap to the “then” part.
Once we accomplish this, we can add complexity to the hypothesized environment one step at a time and see how the conclusions must be amended. This process allows us to derive a reasonably realistic and comprehensible model.
We summarize the simplifying assumptions that lead to the basic version of the CAPM in the following list. The thrust of these assumptions is that we try to ensure that individuals are as alike as possible, with the notable exceptions of initial wealth and risk aversion. We will see that conformity of investor behavior vastly simplifies our analysis.
1. There are many investors, each with an endowment (wealth) that is small compared to the total endowment of all investors. Investors are price-takers, in that they act as though security prices are unaffected by their own trades. This is the usual perfect competition assumption of microeconomics.
2. All investors plan for one identical holding period. This behavior is myopic (short-sighted) in that it ignores everything that might happen after the end of the single-period horizon. Myopic behavior is, in general, suboptimal.
3. Investments are limited to a universe of publicly traded financial assets and to risk-free borrowing or lending arrangements. This assumption rules out investment in nontraded assets such as education (human capital), private enterprises, and governmentally funded assets. It is assumed also that investors may borrow or lend any amount at a fixed, risk-free rate.
4. Investors pay no taxes on returns and no transaction costs (commissions and service charges) on trades in securities. In reality, of course, we know that investors are in different tax brackets and that this may govern the type of assets in which they invest.
5. All investors are rational mean-variance optimizers, meaning that they all use the Markowitz portfolio selection model.
6. All investors analyze securities in the same way and share the same economic view of the world. The result is identical estimates of the probability distribution of future cash flows from investing in the available securities; that is, for any set of security prices, they all derive the same input list to feed into the Markowitz model.
THE CAPITAL ASSET PRICING MODEL

Given a set of security prices and the risk-free interest rate, all investors use the same expected returns and covariance matrix of security returns to generate the efficient frontier and the unique optimal risky portfolio. This assumption is often referred to as homogeneous expectations (同質性預期) or beliefs.
These assumptions represent the “if” of our “what if” analysis. Obviously, they ignore many real-world complexities. With these assumptions, however, we can gain some powerful insights into the nature of equilibrium in security markets.
We can summarize the equilibrium that will prevail in this hypothetical world of securities and investors briefly. The rest of the chapter explains and elaborates on these implications.
1. All investors will choose to hold a portfolio of risky assets in proportions that duplicate representation of the assets in the market portfolio \((M)\) (市場投資組合), which includes all traded assets.
For simplicity, we generally refer to all risky assets as stocks. The proportion of each stock in the market portfolio equals the market value of the stock (price per share multiplied by the number of shares outstanding) divided by the total market value of all stocks.
2. Not only will the market portfolio be on the efficient frontier, but it also will be the tangency portfolio to the optimal capital allocation line (CAL) derived by each and every investor.
As a result, the capital market line (CML), the line from the risk-free rate through the market portfolio, $M$, is also the best attainable capital allocation line. All investors hold $M$ as their optimal risky portfolio, differing only in the amount invested in it versus in the risk-free asset.
3. The risk premium on the market portfolio will be proportional to its risk and the degree of risk aversion of the representative investor. Mathematically,

\[ E(r_M) - r_f = \bar{A} \sigma_M^2 \times 0.01 \]

where \( \sigma_M^2 \) is the variance of the market portfolio and \( \bar{A} \) is the average degree of risk aversion across investors.
Note that because $M$ is the optimal portfolio, which is efficiently diversified across all stocks, $\sigma^2_M$ is the systematic risk of this universe.
The risk premium on individual assets will be proportional to the risk premium on the market portfolio, $M$, and the beta coefficient of the security relative to the market portfolio. Beta measures the extent to which returns on the stock and the market move together.
Formally, beta is defined as

\[ \beta_i = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2} \]

and the risk premium on individual securities is

\[ E(r_i) - r_f = \frac{\text{Cov}(r_i, r_M)}{\sigma_M^2} [E(r_M) - r_f] = \beta_i [E(r_M) - r_f] \]
Why Do All Investors Hold the Market Portfolio?

- What is the market portfolio?
  - When we sum over, or aggregate, the portfolios of all individual investors, lending and borrowing will cancel out (since each lender has a corresponding borrower), and the value of the aggregate risky portfolio will equal the entire wealth of the economy. This is the market portfolio, $M$. 
Why Do All Investors Hold the Market Portfolio?

- The proportion of each stock in the market portfolio equals the market value of the stock (price per share times number of shares outstanding) divided by the sum of the market values of all stocks.
Why Do All Investors Hold the Market Portfolio?

- The CAPM implies that as individuals attempt to optimize their personal portfolios, they each arrive at the same portfolio, with weights on each asset equal to those of the market portfolio.

- Given the assumptions of the previous section, it is easy to see that all investors will desire to hold identical risky portfolios.
Why Do All Investors Hold the Market Portfolio?

- If all investors use identical Markowitz analysis (Assumption 5) applied to the same universe of securities (A. 3) for the same time horizon (A. 2) and use the same input list (A. 6), they all must arrive at the same determination of the optimal risky portfolio, the portfolio on the efficient frontier identified by the tangency line from T-bills to that frontier, as in Figure 9.1.
Figure 9.1
The efficient frontier and the capital market line
Why Do All Investors Hold the Market Portfolio?

- This implies that if the weight of GM stock, for example, in each common risky portfolio is 1%, then GM also will comprise 1% of the market portfolio. The same principle applies to the proportion of any stock in each investor’s risky portfolio. As a result, the optimal risky portfolio of all investors is simply a share of the market portfolio in Figure 9.1.
Why Do All Investors Hold the Market Portfolio?

- Now suppose that the optimal portfolio of our investors does not include the stock of some company, such as Delta Airlines. When all investors avoid Delta stock, the demand is zero, and Delta’s price takes a free fall.
Why Do All Investors Hold the Market Portfolio?

- As Delta stock gets progressively cheaper, it becomes ever more attractive and other stocks look relatively less attractive. Ultimately, Delta reaches a price where it is attractive enough to include in the optimal stock portfolio.
Why Do All Investors Hold the Market Portfolio?

- Such a price adjustment process guarantees that all socks will be included in the optimal portfolio. It shows that all assets have to be included in the market portfolio. The only issue is the price at which investors will be willing to include a stock in their optimal risky portfolio.
Why Do All Investors Hold the Market Portfolio?

- This may seem a roundabout way to derive a simple result: If all investors hold an identical risky portfolio, this portfolio has to be $M$, the market portfolio. Our intention, however, is to demonstrate a connection between this result and its underpinnings, the equilibrating process that is fundamental to security market operation.
The Passive Strategy Is Efficient

- In Chapter 7 we defined the CML (capital market line) as the CAL (capital allocation line) that is constructed from a money market account (or T-bills) and the market portfolio. Perhaps now you can fully appreciate why the CML is an interesting CAL. In the simple world of the CAPM, $M$ is the optimal tangency portfolio on the efficient frontier, as shown in Figure 9.1.
The Passive Strategy Is Efficient

- In this scenario the market portfolio that all investors hold is based on the common input list, thereby incorporating all relevant information about the universe of securities. This means that investors can skip the trouble of doing specific analysis and obtain an efficient portfolio simply by holding the market portfolio.
The Passive Strategy Is Efficient

- Of course, if everyone were to follow this strategy, no one would perform security analysis and this result would no longer hold. We discuss this issue in greater depth in Chapter 12 on market efficiency.
The Passive Strategy Is Efficient

- Thus the passive strategy of investing in a market index portfolio is efficient. For this reason, we sometimes call this result a mutual fund theorem (共同基金理論). The mutual fund theorem is another incarnation of the separation property discussed in Chapter 8.
The Passive Strategy Is Efficient

- Assuming that all investors choose to hold a market index mutual fund, we can separate portfolio selection into two components—a technological problem, creation of mutual funds by professional managers—and a personal problem that depends on an investor’s risk aversion, allocation of the complete portfolio between the mutual fund and risk-free assets.
The Passive Strategy Is Efficient

- In reality, different investment managers do create risky portfolios that differ from the market index. We attribute this in part to the use of different input lists in the formation of the optimal risky portfolio. Nevertheless, the practical significance of the mutual fund theorem is that a passive investor may view the market index as a reasonable first approximation to an efficient risky portfolio.
The Risk Premium of the Market Portfolio

- In Chapter 7 we discussed how individual investors go about deciding how much to invest in the risky portfolio. Returning now to the decision of how much to invest in portfolio $M$ versus in the risk-free asset, what can we deduce about the equilibrium risk premium of portfolio $M$?
The Risk Premium of the Market Portfolio

- We asserted earlier that the equilibrium risk premium on the market portfolio, $E(r_M) - r_f$, will be proportional to the average degree of risk aversion of the investor population and the risk of the market portfolio, $\sigma^2_M$. Now we can explain this result.
The Risk Premium of the Market Portfolio

- Recall that each individual investor chooses a proportion $y$, allocated to the optimal portfolio $M$, such that

$$y = \frac{E(r_M) - r_f}{.01 \times A \sigma^2_M} \quad (9.1)$$
In the simplified CAPM economy, risk-free investments involve borrowing and lending among investors. Any borrowing position must be offset by the lending position of the creditor. This means that net borrowing and lending across all investors must be zero, and in consequence the average position in the risky portfolio is 100%, or $\bar{y} = 1$. 
The Risk Premium of the Market Portfolio

- Setting $\bar{y} = 1$ in equation 9.1 and rearranging, we find that the risk premium on the market portfolio is related to its variance by the average degree of risk aversion:

$$E(r_M) - r_f = 0.01 \times \bar{A} \sigma_M^2 \quad (9.2)$$
Expected Returns on Individual Securities

- The CAPM is built on the insight that the appropriate risk premium on an asset will be determined by its contribution to the risk of investors’ overall portfolios. Portfolio risk is what matters to investors and is what governs the risk premiums they demand.
Expected Returns on Individual Securities

- Remember that all investors use the same input list, that is, the same estimates of expected returns, variances, and covariances.
Expected Returns on Individual Securities

- We saw in Chapter 8 that these covariances can be arranged in a covariance matrix, so that the entry in the fifth row and third column, for example, would be the covariance between the rates of return on the fifth and third securities.
Expected Returns on Individual Securities

- Each diagonal entry of the matrix is the covariance of one security’s return with itself, which is simply the variance of that security. We will consider the construction of the input list a bit later. For now we take it as given.
Expected Returns on Individual Securities

- Suppose, for example, that we want to gauge the portfolio risk of GM stock. We measure the contribution to the risk of the overall portfolio from holding GM stock by its covariance with the market portfolio.
To see why this is so, let us look again at the way the variance of the market portfolio is calculated. To calculate the variance of the market portfolio, we use the bordered covariance matrix with the market portfolio weights, as discussed in Chapter 8. We highlight GM in this depiction of the $n$ stocks in the market portfolio.
<table>
<thead>
<tr>
<th>Portfolio Weights</th>
<th>( w_1 )</th>
<th>( w_2 )</th>
<th>( \ldots )</th>
<th>( w_{GM} )</th>
<th>( \ldots )</th>
<th>( w_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 )</td>
<td>( \text{Cov}(r_1, r_1) )</td>
<td>( \text{Cov}(r_1, r_2) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_1, r_{GM}) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_1, r_n) )</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>( \text{Cov}(r_2, r_1) )</td>
<td>( \text{Cov}(r_2, r_2) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_2, r_{GM}) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_2, r_n) )</td>
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<td>( \vdots )</td>
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<tr>
<td>( w_{GM} )</td>
<td>( \text{Cov}(r_{GM}, r_1) )</td>
<td>( \text{Cov}(r_{GM}, r_2) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_{GM}, r_{GM}) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_{GM}, r_n) )</td>
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</tr>
<tr>
<td>( w_n )</td>
<td>( \text{Cov}(r_n, r_1) )</td>
<td>( \text{Cov}(r_n, r_2) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_n, r_{GM}) )</td>
<td>( \ldots )</td>
<td>( \text{Cov}(r_n, r_n) )</td>
</tr>
</tbody>
</table>
Expected Returns on Individual Securities

- Recall that we calculate the variance of the portfolio by summing over all the elements of the covariance matrix, first multiplying each element by the portfolio weights from the row and the column.
Expected Returns on Individual Securities

- The contribution of one stock to portfolio variance therefore can be expressed as the sum of all the covariance terms in the row corresponding to the stock, where each covariance is first multiplied by both the stock’s weight from its row and the weight from its column.
Expected Returns on Individual Securities

- For example, the contribution of GM’s stock to the variance of the market portfolio is

\[ w_{GM} \left[ w_1 \text{Cov}(r_1, r_{GM}) + w_2 \text{Cov}(r_2, r_{GM}) + \ldots + w_{GM} \text{Cov}(r_{GM}, r_{GM}) + \ldots + w_n \text{Cov}(r_n, r_{GM}) \right] \]  

(9.3)
Expected Returns on Individual Securities

- Equation 9.3 provides a clue about the respective roles of variance and covariance in determining asset risk. When there are many stocks in the economy, there will be many more covariance terms than variance terms. Consequently, the covariance of a particular stock with all other stocks will dominate that stock’s contribution to total portfolio risk.
Expected Returns on Individual Securities

- We may summarize the terms in square brackets in equation 9.3 simply as the covariance of GM with the market portfolio. In other words, we can best measure the stock’s contribution to the risk of the market portfolio by its covariance with that portfolio:

\[ \text{GM’s contribution to variance} = w_{GM} \text{Cov}(r_{GM}, r_M) \]
Expected Returns on Individual Securities

- This should not surprise us. For example, if the covariance between GM and the rest of the market is negative, then GM makes a “negative contribution” to portfolio risk: By providing returns that move inversely with the rest of the market, GM stabilizes the return on the overall portfolio.
Expected Returns on Individual Securities

- If the covariance is positive, GM makes a positive contribution to overall portfolio risk because its returns amplify swings in the rest of the portfolio.
Expected Returns on Individual Securities

To demonstrate this more rigorously, note that the rate of return on the market portfolio may be written as

\[ r_M = \sum_{k=1}^{n} w_k r_k \]
Expected Returns on Individual Securities

Therefore, the covariance of the return on GM with the market portfolio is

\[
\text{Cov}(r_{GM}, r_M) = \text{Cov} \left( r_{GM}, \sum_{k=1}^{n} w_k r_k \right)
\]

\[
= \sum_{k=1}^{n} w_k \text{Cov}(r_{GM}, r_k)
\]

(9.4)
Expected Returns on Individual Securities

Notice that the last term of equation 9.4 is precisely the same as the term in brackets in equation 9.3. Therefore, equation 9.3, which is the contribution of GM to the variance of the market portfolio, may be simplified to $w_{GM} \text{Cov}(r_{GM}, r_M)$. 
Expected Returns on Individual Securities

- We also observe that the contribution of our holding of GM to the risk premium of the market portfolio is $w_{GM}[E(r_M) - r_f]$. 
Therefore, the reward-to-risk ratio for investments in GM can be expressed as

\[
\frac{w_{GM} \left[ E(r_{GM}) - r_f \right]}{w_{GM} \text{Cov}(r_{GM}, r_M)} = \frac{E(r_{GM}) - r_f}{\text{Cov}(r_{GM}, r_M)}
\]
Expected Returns on Individual Securities

- The market portfolio is the tangency (efficient mean-variance) portfolio. The reward-to-risk ratio for investment in the market portfolio is

\[
\text{Market risk premium} = \frac{E(r_M) - r_f}{\sigma^2_M} \quad (9.5)
\]
Expected Returns on Individual Securities

The ratio in equation 9.5 is often called the market price of risk (風險的市場價格) because it quantifies the extra return that investors demand to bear portfolio risk. Notice that for components of the efficient portfolio, such as shares of GM, we measure risk as the contribution to portfolio variance. In contrast, for the efficient portfolio itself, its variance is the appropriate measure of risk.
Expected Returns on Individual Securities

- A basic principle of equilibrium is that all investments should offer the same reward-to-risk ratio. If the ratio were better for one investment than another, investors would rearrange their portfolios, tilting toward the alternative with the better trade-off and shying away from the other.
Expected Returns on Individual Securities

Such activity would impart pressure on security prices until the ratios were equalized. Therefore we conclude that the reward-to-risk ratios of GM and the market portfolio should be equal:

\[ \frac{E(r_{GM}) - r_f}{\text{Cov}(r_{GM}, r_M)} = \frac{E(r_M) - r_f}{\sigma_M^2} \]  

(9.6)
Expected Returns on Individual Securities

To determine the fair risk premium of GM stock, we rearrange equation (9.6) slightly to obtain

\[ E(r_{GM}) - r_f = \frac{\text{Cov}(r_{GM}, r_M)}{\sigma_M^2} \left[ E(r_M) - r_f \right] \]  

(9.7)
Expected Returns on Individual Securities

- The ratio measures the contribution of GM stock to the variance of the market portfolio as a fraction of the total variance of the market portfolio. The ratio is called beta and is denoted by $\beta$. Using this measure, we can restate equation 9.7 as

$$E(r_{GM}) = r_f + \beta_{GM}[E(r_M) - r_f]$$  \hspace{1cm} (9.8)
Expected Returns on Individual Securities

- This expected return–beta relationship of equation 9.8 is the most familiar expression of the CAPM to practitioners. We will have a lot more to say about the expected return–beta relationship shortly.
Expected Returns on Individual Securities

- We see now why the assumptions that made individuals act similarly are so useful. If everyone holds an identical risky portfolio, then everyone will find that the beta of each asset with the market portfolio equals the asset’s beta with his or her own risky portfolio. Hence everyone will agree on the appropriate risk premium for each asset.
Does the fact that few real-life investors actually hold the market portfolio imply that the CAPM is of no practical importance?

Not necessarily. Recall from Chapter 8 that reasonably well-diversified portfolios shed firm-specific risk and are left with mostly systematic or market risk.
Expected Returns on Individual Securities

- Even if one does not hold the precise market portfolio, a well-diversified portfolio will be so very highly correlated with the market that a stock’s beta relative to the market will still be a useful risk measure.
In fact, several authors have shown that modified versions of the CAPM will hold true even if we consider differences among individuals leading them to hold different portfolios.
Expected Returns on Individual Securities

- For example, Brennan examined the impact of differences in investors’ personal tax rates on market equilibrium, and Mayers looked at the impact of nontraded assets such as human capital (earning power).
  - Both found that although the market portfolio is no longer each investor’s optimal risky portfolio, the expected return–beta relationship should still hold in a somewhat modified form.
If the expected return–beta relationship holds for any individual asset, it must hold for any combination of assets. Suppose that some portfolio $P$ has weight $w_k$ for stock $k$, where $k$ takes on values 1,..., $n$. Writing out the CAPM equation 9.8 for each stock, and multiplying each equation by the weight of the stock in the portfolio, we obtain these equations, one for each stock: (next slide)
Expected Returns on Individual Securities

\[ w_1 E(r_1) = w_1 r_f + w_1 \beta_1 [E(r_M) - r_f] \]
\[ + w_2 E(r_2) = w_2 r_f + w_2 \beta_2 [E(r_M) - r_f] \]
\[ + \ldots = \ldots \]
\[ + w_n E(r_n) = w_n r_f + w_n \beta_n [E(r_M) - r_f] \]
\[ \Rightarrow E(r_p) = r_f + \beta_p [E(r_M) - r_f] \]
Expected Returns on Individual Securities

- Summing each column shows that the CAPM holds for the overall portfolio because $E(r_p) = \sum w_k E(r_k)$ is the expected return on the portfolio, and $\beta_p = \sum w_k \beta_k$ is the portfolio beta. Incidentally, this result has to be true for the market portfolio itself,

$$E(r_M) = r_f + \beta_M [E(r_M) - r_f]$$
Expected Returns on Individual Securities

Indeed, this is a tautology because $\beta_M = 1$, as we can verify by noting that

$$
\beta_M = \frac{\text{Cov}(r_M, r_M)}{\sigma_M^2} = \frac{\sigma_M^2}{\sigma_M^2} = 1
$$
Expected Returns on Individual Securities

This also establishes $l$ as the weighted-average value of beta across all assets. If the market beta is $l$, and the market is a portfolio of all assets in the economy, the weighted-average beta of all assets must be $l$. 
Expected Returns on Individual Securities

- Hence betas greater than 1 are considered aggressive in that investment in high-beta stocks entails above-average sensitivity to market swings. Betas below 1 can be described as defensive.
Expected Returns on Individual Securities

- A word of caution: We are all accustomed to hearing that well-managed firms will provide high rates of return. We agree this is true if one measures the firm’s return on investments in plant and equipment. The CAPM, however, predicts returns on investments in the securities of the firm.
Expected Returns on Individual Securities

- Let us say that everyone knows a firm is well run. Its stock price will therefore be bid up, and consequently returns to stockholders who buy at those high prices will not be excessive.
Expected Returns on Individual Securities

- Security prices, in other words, already reflect public information about a firm’s prospects; therefore only the risk of the company (as measured by beta in the context of the CAPM) should affect expected returns. In a rational market investors receive high expected returns only if they are willing to bear risk.
Expected Returns on Individual Securities

- Of course, investors do not directly observe or determine expected returns on securities. Rather, they observe security prices and bid those prices up or down. Expected rates of return are determined by the prices investors must pay compared to the cash flows those investments might garner.
We can view the expected return–beta relationship as a reward–risk equation. The beta of a security is the appropriate measure of its risk because beta is proportional to the risk that the security contributes to the optimal risky portfolio.
Risk-averse investors measure the risk of the optimal risky portfolio by its variance. In this world we would expect the reward, or the risk premium on individual assets, to depend on the contribution of the individual asset to the risk of the portfolio.
The Security Market Line

- The beta of a stock measures the stock’s contribution to the variance of the market portfolio. Hence we expect, for any asset or portfolio, the required risk premium to be a function of beta.
The Security Market Line

- The CAPM confirms this intuition, stating further that the security’s risk premium is directly proportional to both the beta and the risk premium of the market portfolio; that is, the risk premium equals $\beta [E(r_M) - r_f]$. 
The Security Market Line

- The expected return–beta relationship can be portrayed graphically as the security market line (SML) in Figure 9.2. Because the market beta is 1, the slope is the risk premium of the market portfolio. At the point on the horizontal axis where $\beta = 1$ (which is the market portfolio’s beta) we can read off the vertical axis the expected return on the market portfolio.
Figure 9.2
The security market line

\[ E(r) \]

\[ E(r_m) - r_f = \text{Slope of SML} \]

\[ \beta_m = 1.0 \]

\[ r_f \]

\[ 1 \]
The Security Market Line

- It is useful to compare the security market line to the capital market line.
  - The CML graphs the risk premiums of *efficient* portfolios (i.e., portfolios composed of the market and the risk-free asset) as a function of portfolio standard deviation. This is appropriate because standard deviation is a valid measure of risk for efficiently diversified portfolios that are candidates for an investor’s overall portfolio.
The Security Market Line

- The SML, in contrast, graphs *individual asset* risk premiums as a function of asset risk. The relevant measure of risk for individual assets held as parts of well-diversified portfolios is not the asset’s standard deviation or variance; it is, instead, the contribution of the asset to the portfolio variance, which we measure by the asset’s beta. The SML is valid for both efficient portfolios and individual assets.
The Security Market Line

- The security market line provides a benchmark for the evaluation of investment performance. Given the risk of an investment, as measured by its beta, the SML provides the required rate of return necessary to compensate investors for both risk as well as the time value of money.
Because the security market line is the graphic representation of the expected return–beta relationship, “fairly priced” assets plot exactly on the SML; that is, their expected returns are commensurate with their risk. Given the assumptions we made at the start of this section, all securities must lie on the SML in market equilibrium.
Nevertheless, we see here how the CAPM may be of use in the money-management industry. Suppose that the SML relation is used as a benchmark to assess the fair expected return on a risky asset. Then security analysis is performed to calculate the return actually expected.
If a stock is perceived to be a good buy, or underpriced, it will provide an expected return in excess of the fair return stipulated by the SML. Underpriced stocks therefore plot above the SML: Given their betas, their expected returns are greater than dictated by the CAPM. Overpriced stocks plot below the SML.
The Security Market Line

- The difference between the fair and actually expected rates of return on a stock is called the stock’s **alpha**, denoted \( \alpha \).
  
  - For example, if the market return is expected to be 14%, a stock has a beta of 1.2, and the T-bill rate is 6%, the SML would predict an expected return on the stock of \( 6\% + 1.2(14\% - 6\%) = 15.6\% \). If one believed the stock would provide an expected return of 17%, the implied alpha would be 1.4% (see Figure 9.3).
Figure 9.3
The SML and a positive-alpha stock
One might say that security analysis (which we treat in Part 5) is about uncovering securities with nonzero alphas. This analysis suggests that the starting point of portfolio management can be a passive market-index portfolio. The portfolio manager will then increase the weights of securities with positive alphas and decrease the weights of securities with negative alphas.
The CAPM is also useful in capital budgeting decisions. For a firm considering a new project, the CAPM can provide the required rate of return that the project needs to yield, based on its beta, to be acceptable to investors. Managers can use the CAPM to obtain this cutoff internal rate of return (IRR), or “hurdle rate” for the project.
EXAMPLE 9.1: Using the CAPM

- Yet another use of the CAPM is in utility rate-making cases. In this case the issue is the rate of return that a regulated utility should be allowed to earn on its investment in plant and equipment.
EXAMPLE 9.1: Using the CAPM

- Suppose that the equityholders have invested $100 million in the firm and that the beta of the equity is .6. If the T-bill rate is 6% and the market risk premium is 8%, then the fair profits to the firm would be assessed as 6% + .6(8%) = 10.8% of the $100 million investment, or $10.8 million. The firm would be allowed to set prices at a level expected to generate these profits.
EXTENSIONS OF THE CAPM

- The assumptions that allowed Sharpe to derive the simple version of the CAPM are admittedly unrealistic. Financial economists have been at work ever since the CAPM was devised to extend the model to more realistic scenarios.
EXTENSIONS OF THE CAPM

- There are two classes of extensions to the simple version of the CAPM.
  - The first attempts to relax the assumptions that we outlined at the outset of the chapter.
The second acknowledges the fact that investors worry about sources of risk other than the uncertain value of their securities, such as unexpected changes in relative prices of consumer goods. This idea involves the introduction of additional risk factors besides security returns, and we discuss it further in Chapter 11.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- The CAPM is predicated on the assumption that all investors share an identical input list that they feed into the Markowitz algorithm. Thus all investors agree on the location of the efficient (minimum-variance) frontier, where each portfolio has the lowest variance among all feasible portfolios at a target expected rate of return.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- When all investors can borrow and lend at the safe rate, $r_f$, all agree on the optimal tangency portfolio and choose to hold a share of the market portfolio.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- However, when borrowing is restricted, as it is for many financial institutions, or when the borrowing rate is higher than the lending rate because borrowers pay a default premium, the market portfolio is no longer the common optimal portfolio for all investors.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- When investors no longer can borrow at a common risk-free rate, they may choose risky portfolios from the entire set of efficient frontier portfolios according to how much risk they choose to bear. The market is no longer the common optimal portfolio.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- In fact, with investors choosing different portfolios, it is no longer obvious whether the market portfolio, which is the aggregate of all investors’ portfolios, will even be on the efficient frontier.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- If the market portfolio is no longer mean-variance efficient, then the expected return–beta relationship of the CAPM will no longer characterize market equilibrium.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- An equilibrium expected return–beta relationship in the case of restrictions on risk-free investments has been developed by Fischer Black. Black’s model is fairly difficult and requires a good deal of facility with mathematics. Therefore, we will satisfy ourselves with a sketch of Black’s argument and spend more time with its implications.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- Black’s model of the CAPM in the absence of a risk-free asset rests on the three following properties of mean-variance efficient portfolios:
  - Property 1. Any portfolio constructed by combining efficient portfolios is itself on the efficient frontier.
Property 2. Every portfolio on the efficient frontier has a “companion” portfolio on the bottom half (the inefficient part) of the minimum-variance frontier with which it is uncorrelated. Because the portfolios are uncorrelated, the companion portfolio is referred to as the zero-beta portfolio of the efficient portfolio.
The expected return of an efficient portfolio’s zero-beta companion portfolio can be derived by the following graphical procedure. From any efficient portfolio such as $P$ in Figure 9.4 on page 296 draw a tangency line to the vertical axis. The intercept will be the expected return on portfolio $P$’s zero-beta companion portfolio, denoted $Z(P)$. 
Figure 9.4
Efficient portfolios and their zero-beta companions
The horizontal line from the intercept to the minimum-variance frontier identifies the standard deviation of the zero-beta portfolio. Notice in Figure 9.4 that different efficient portfolios such as $P$ and $Q$ have different zero-beta companions.
These tangency lines are helpful constructs only. They do \textit{not} signify that one can invest in portfolios with expected return-standard deviation pairs along the line. That would be possible only by mixing a risk-free asset with the tangency portfolio. In this case, however, we assume that risk-free assets are not available to investors.
The CAPM with Restricted Borrowing: The Zero-Beta Model

Property 3. The expected return of any asset can be expressed as an exact, linear function of the expected return on any two frontier portfolios. Consider, for example, the minimum-variance frontier portfolios $P$ and $Q$. 
The CAPM with Restricted Borrowing: The Zero-Beta Model

Black showed that the expected return on any asset $i$ can be expressed as

$$E(r_i) = E(r_Q) + [E(r_p) - E(r_i)] \frac{\text{Cov}(r_i, r_p) - \text{Cov}(r_p, r_Q)}{\sigma_p^2 - \text{Cov}(r_p, r_Q)}$$

(9.9)

Note that Property 3 has nothing to do with market equilibrium. It is a purely mathematical property relating frontier portfolios and individual securities.
With these three properties, the Black model can be applied to any of several variations: no risk-free asset at all, risk-free lending but no risk-free borrowing, and borrowing at a rate higher than $r_f$. We show here how the model works for the case with risk-free lending but no borrowing.
Imagine an economy with only two investors, one relatively risk averse and one risk tolerant. The risk-averse investor will choose a portfolio on the CAL supported by portfolio \( T \) in Figure 9.5, that is, he will mix portfolio \( T \) with lending at the risk-free rate. \( T \) is the tangency portfolio on the efficient frontier from the risk-free lending rate, \( r_f \).
Figure 9.5
Capital market equilibrium with no borrowing

[Diagram showing capital market equilibrium with no borrowing]
The CAPM with Restricted Borrowing: The Zero-Beta Model

- The risk-tolerant investor is willing to accept more risk to earn a higher-risk premium; she will choose portfolio $S$. This portfolio lies along the efficient frontier with higher risk and return than portfolio $T$. 

The CAPM with Restricted Borrowing: The Zero-Beta Model

- The aggregate risky portfolio (i.e., the market portfolio, $M$) will be a combination of $T$ and $S$, with weights determined by the relative wealth and degrees of risk aversion of the two investors. Since $T$ and $S$ are each on the efficient frontier, so is $M$ (from Property 1).
From Property 2, $M$ has a companion zero-beta portfolio on the minimum-variance frontier, $Z(M)$, shown in Figure 9.5. Moreover, by Property 3 we can express the return on any security in terms of $M$ and $Z(M)$ as in equation 9.9.
The CAPM with Restricted Borrowing: The Zero-Beta Model

- But, since by construction $\text{Cov}[r_M, r_{Z(M)}] = 0$, the expression simplifies to

$$E(r_i) = E[r_{Z(M)}] + E[r_M - r_{Z(M)}] \frac{\text{Cov}(r_i, r_M)}{\sigma^2_M} \quad (9.10)$$

where $P$ from equation 9.9 has been replaced by $M$ and $Q$ has been replaced by $Z(M)$. 
The CAPM with Restricted Borrowing: The Zero-Beta Model

Equation 9.10 may be interpreted as a variant of the simple CAPM, in which $r_f$ has been replaced with $E[r_{Z(M)}]$. 
The more realistic scenario, where investors lend at the risk-free rate and borrow at a higher rate, was considered in Chapter 8. The same arguments that we have just employed can also be used to establish the zero-beta CAPM in this situation.
One of the restrictive assumptions for the simple version of the CAPM is that investors are myopic—they plan for one common holding period.

Investors actually may be concerned with a lifetime consumption plan and a desire to leave a bequest to children.
Lifetime Consumption and the CAPM

- Consumption plans that are feasible for them depend on current wealth and future rates of return on the investment portfolio. These investors will want to rebalance their portfolios as often as required by changes in wealth.
Lifetime Consumption and the CAPM

- However, Eugene Fama showed that, even if we extend our analysis to a multiperiod setting, the single-period CAPM still may be appropriate. The key assumptions that Fama used to replace myopic planning horizons are that investor preferences are unchanging over time and the risk-free interest rate and probability distribution of security returns do not change unpredictably over time.
Lifetime Consumption and the CAPM

- Of course, this latter assumption is also unrealistic. A variant of the CAPM that allows for such unpredictability is presented in Chapter 11.
Liquidity refers to the cost and ease with which an asset can be converted into cash, that is, sold. Traders have long recognized the importance of liquidity, and some evidence suggests that illiquidity can reduce market prices substantially.

- For example, one study finds that market discounts on closely held (and therefore nontraded) firms can exceed 30%.
Several studies show that liquidity plays an important role in explaining rates of return on financial assets. Chordia, Roll, and Subrahmanyam (CRS) find commonality across stocks in the variable cost of liquidity:

- quoted spreads
- quoted depth
- effective spreads covary with the market and industrywide liquidity
THE CAPM AND LIQUIDITY

- Hence, liquidity risk is systematic and therefore difficult to diversify. We believe that liquidity will become an important part of standard valuation, and therefore present here a simplified version of the CRS model.
THE CAPM AND LIQUIDITY

- Recall Assumption 4 of the CAPM, that all trading is costless. In reality, no security is perfectly liquid, in that all trades involve some transaction cost.
Investors prefer more liquid assets with lower transaction costs, so it should not surprise us to find that all else equal, relatively illiquid assets trade at lower prices or, equivalently, that the expected return on illiquid assets must be higher. Therefore, an \textit{illiquidity premium} (流動性溢酬) must be impounded into the price of each asset.
THE CAPM AND LIQUIDITY

- We start with the simplest case, in which we ignore systematic risk. Imagine a world with a large number of uncorrelated securities. Because the securities are uncorrelated, well-diversified portfolios of these securities will have standard deviations near zero and the market portfolio will be virtually as safe as the risk-free asset.
In this case, the market risk premium will be zero. Therefore, despite the fact that the beta of each security is 1.0, the expected rate of return on all securities will equal the risk-free rate, which we will take to be the T-bill rate.
THE CAPM AND LIQUIDITY

- Assume that investors know in advance for how long they intend to hold their portfolios, and suppose that there are $n$ types of investors, grouped by investment horizon. Type 1 investors intend to liquidate their portfolios in one period, Type 2 investors in two periods, and so on, until the longest-horizon investors (Type $n$) intend to hold their portfolios for $n$ periods.
THE CAPM AND LIQUIDITY

- We assume that there are only two classes of securities: liquid and illiquid. The liquidation cost of a class L (more liquid) stock to an investor with a horizon of $h$ years (a Type $h$ investor) will reduce the per-period rate of return by $c_L/h\%$. 
THE CAPM AND LIQUIDITY

- For example, if the combination of commissions and the bid-asked spread on a security resulted in a liquidation cost of 10%, then the per-period rate of return for an investor who holds stock for 5 years would be reduced by approximately 2% per year, whereas the return on a 10-year investment would fall by only 1% per year.
THE CAPM AND LIQUIDITY

- Class I (illiquid) assets have higher liquidation costs that reduce the per-period return by $c_1/h\%$, where $c_1$ is greater than $c_L$.
- Therefore, if you intend to hold a class L security for $h$ periods, your expected rate of return net of transaction costs is $E(r_L) - c_L/h$.
- There is no liquidation cost on T-bills.
The following table presents the expected return investors would realize from the risk-free asset and class L and class I stock portfolios *assuming* that the simple CAPM is correct and all securities have an expected return of $r$: (next slide)
<table>
<thead>
<tr>
<th>Investor Type</th>
<th>Risk-Free</th>
<th>Class L</th>
<th>Class L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross rate of return:</td>
<td>$r$</td>
<td>$r$</td>
<td>$r$</td>
</tr>
<tr>
<td>One-period liquidation cost:</td>
<td>0</td>
<td>$c_L$</td>
<td>$c_1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Investor Type</th>
<th>Net Rate of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$r - c_L$</td>
</tr>
<tr>
<td>2</td>
<td>$r - c_L/2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$r - c_L/n$</td>
</tr>
</tbody>
</table>
THE CAPM AND LIQUIDITY

- These net rates of return would be inconsistent with a market in equilibrium, because with equal gross rates of return all investors would prefer to invest in zero-transaction-cost T-bills. As a result, both class L and class I stock prices must fall, causing their expected returns to rise until investors are willing to hold these shares.
Suppose, therefore, that each gross return is higher by some fraction of liquidation cost. Specifically, assume that the gross expected return on class \( L \) stock is \( r + xc_L \) and that of class \( I \) stocks is \( r + yc_I \). The net rate of return on class \( L \) stocks to an investor with a horizon of \( h \) will be \( (r + xc_L) - c_L/h = r + c_L(x - 1/h) \). In general, the rates of return to investors will be: (next slide)
<table>
<thead>
<tr>
<th>Asset:</th>
<th>Risk-Free</th>
<th>Class L</th>
<th>Class I</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gross rate of return:</td>
<td>$r$</td>
<td>$r + xc_L$</td>
<td>$r + yc_I$</td>
</tr>
<tr>
<td>One-period liquidation cost:</td>
<td>0</td>
<td>$c_L$</td>
<td>$c_I$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Investor Type</th>
<th>Net Rate of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$r$</td>
</tr>
<tr>
<td>2</td>
<td>$r$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$n$</td>
<td>$r$</td>
</tr>
</tbody>
</table>
Notice that the liquidation cost has a greater impact on per-period returns for shorter-term investors. This is because the cost is amortized over fewer periods. As the horizon becomes very large, the per-period impact of the transaction cost approaches zero and the net rate of return approaches the gross rate.
Figure 9.6 graphs the net rate of return on the three asset classes for investors of differing horizons.

The more illiquid stock has the lowest net rate of return for very short investment horizons because of its large liquidation costs.
Figure 9.6
Net returns as a function of investment horizon

\[ r_i = r + yC_i \]

\[ r_L = r + xC_L \]

- T-Bills Dominate
- Class L Stocks Dominate
- Class I Stocks Dominate
However, in equilibrium, the illiquid stock must be priced at a level that offers a rate of return high enough to induce some investors to hold it, implying that its gross rate of return must be higher than that of the more liquid stock. Therefore, for long enough investment horizons, the net return on class I stocks will exceed that on class L stocks.
Both stock classes underperform T-bills for very short investment horizons, because the transactions costs then have the largest per-period impact. Ultimately, however, because the *gross* rate of return of stocks exceeds $r$, for a sufficiently long investment horizon, the more liquid stocks in class L will dominate bills. The threshold horizon can be read from Figure 9.6 as $h_{rL}$. 
Anyone with a horizon that exceeds $h_{rL}$ will prefer class L stocks to T-bills. Those with horizons below $h_{rL}$ will choose bills. For even longer horizons, because $c_I$ exceeds $c_L$, the net rate of return on relatively illiquid class I stocks will exceed that on class L stocks.
Therefore, investors with horizons greater than $h_{LI}$ will specialize in the most illiquid stocks with the highest gross rate of return. These investors are harmed least by the effect of trading costs.
Now we can determine equilibrium illiquidity premiums. For the marginal investor with horizon $h_{LI}$, the net return from class I and L stocks is the same. Therefore,

$$r + c_L(x - 1/h_{LI}) = r + c_I(y - 1/h_{LI})$$
THE CAPM AND LIQUIDITY

We can use this equation to solve for the relationship between $x$ and $y$ as follows:

$$y = \frac{1}{h_{LI}} + \frac{c_L}{c_I} \left( x - \frac{1}{h_{LI}} \right)$$
The expected gross return on illiquid stocks is then

\[ r_1 = r + c_I y = r + \frac{c_I}{h_{LI}} + c_L \left( x - \frac{1}{h_{LI}} \right) \]

\[ = r + c_L x + \frac{1}{h_{LI}}(c_I - c_L) \]

(9.11)
THE CAPM AND LIQUIDITY

Recalling that the expected gross return on class L stocks is $r_L = r + c_L x$, we conclude that the illiquidity premium of class I versus class L stock is

$$r_I - r_L = \frac{1}{h_{LI}} (c_I - c_L) \quad (9.12)$$
Similarly, we can derive the liquidity premium of class L stocks over T-bills. Here, the marginal investor who is indifferent between bills and class L stocks will have investment horizon $h_{rL}$ and a net rate of return just equal to $r$. 
Therefore, \( r + c_L(x - 1/h_{rL}) = r \), implying that \( x = 1/h_{rL} \), and the liquidity premium of class \( L \) stocks must be \( xc_L = c_L/h_{rL} \). Therefore,

\[
r_L - r = \frac{1}{h_{rL}} c_L \quad \text{(9.13)}
\]
THE CAPM AND LIQUIDITY

- There are two lessons to be learned from this analysis.
  - First, as predicted, equilibrium expected rates of return are bid up to compensate for transaction costs, as demonstrated by equations 9.12 and 9.13.
Second, the illiquidity premium is not a linear function of transaction costs. In fact, the incremental illiquidity premium steadily declines as transaction costs increase. To see that this is so, suppose that $c_L$ is 1% and $c_I - c_L$ is also 1%. Therefore, the transaction cost increases by 1% as you move out of bills into the more liquid stock class, and by another 1% as you move into the illiquid stock class.
Equation 9.13 shows that the illiquidity premium of class L stocks over no-transaction-cost bills is then $1/h_{rL}$, and equation 9.12 shows that the illiquidity premium of class I over class L socks is $1/h_{LI}$. But $h_{LI}$ exceeds $h_{rL}$ (see Figure 9.5), so we conclude that the incremental effect of illiquidity declines as we move into ever more illiquid assets.
THE CAPM AND LIQUIDITY

The reason for this last result is simple. Recall that investors will self-select into different asset classes, with longer-term investors holding assets with the highest gross return but that are the most illiquid. For these investors, the effect of illiquidity is less costly because trading costs can be amortized over a longer horizon.
Therefore, as these costs increase, the investment horizon associated with the holders of these assets also increases, which mitigates the impact on the required gross rate of return.
Our analysis so far has focused on the case of uncorrelated assets, allowing us to ignore issues of systematic risk. This special case turns out to be easy to generalize.
THE CAPM AND LIQUIDITY

- If we were to allow for correlation among assets due to common systematic risk factors, we would find that the illiquidity premium is simply additive to the risk premium of the usual CAPM.
Therefore, we can generalize the CAPM expected return–beta relationship to include a liquidity effect as follows:

\[ E(r_i) - r_f = \beta_i [E(r_M) - r_f] + f(c_i) \]

where \( f(c_i) \) is a function of trading costs that measures the effect of the illiquidity premium given the trading costs of security \( i \).
THE CAPM AND LIQUIDITY

• We have seen that $f(c_i)$ is increasing in $c_i$ but at a decreasing rate. The usual CAPM equation is modified because each investor’s optimal portfolio is now affected by liquidation cost as well as risk–return considerations.
The model can be generalized in other ways as well.

- For example, even if investors do not know their investment horizon for certain, as long as investors do not perceive a connection between unexpected needs to liquidate investments and security returns, the implications of the model are essentially unchanged, with expected horizons replacing actual horizons in equations 9.12 and 9.13.
THE CAPM AND LIQUIDITY

- Amihud and Mendelson provided a considerable amount of empirical evidence that liquidity has a substantial impact on gross stock returns. For a preview of the quantitative significance of the illiquidity effect, we examine Figure 9.7, which is derived from their study.
Figure 9.7
The relationship between illiquidity and average returns

THE CAPM AND LIQUIDITY

- Figure 9.7 shows that average monthly returns over the 1961-1980 period rose from .35% for the group of stocks with the lowest bid-asked spread (the most liquid stocks) to 1.024% for the highest-spread stocks.
This is an annualized differential of about 8%, nearly equal to the historical average risk premium on the S&P 500 index! Moreover, as their model predicts, the effect of the spread on average monthly returns is nonlinear, with a curve that flattens out as spreads increase.