CHAPTER 11

ARBITRAGE PRICING THEORY AND MULTIFACTOR MODELS OF RISK AND RETURN
Multifactor Models: An Overview
Arbitrage Pricing Theory
Individual Assets and the APT
A Multifactor APT
Where Should We Look for Factors?
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The index model introduced in Chapter 10 gave us a way of decomposing stock variability into market or systematic risk, due largely to macroeconomic events, versus firm-specific or idiosyncratic effects that can be diversified in large portfolios.
MULTIFACTOR MODELS: AN OVERVIEW

- In the index model, the return on the market portfolio summarized the broad impact of macro factors. Sometimes, however, rather than using a market proxy, it is more useful to focus directly on the ultimate sources of risk. This can be useful in risk assessment, for example, when measuring one’s exposures to particular sources of uncertainty.
MULTIFACTOR MODELS: AN OVERVIEW

- Factor models are tools that allow us to describe and quantify the different factors that affect the rate of return on a security during any time period.
Factor Models of Security Returns

- To illustrate, we will start by examining a single-factor model like the one introduced in Chapter 10. As noted there, uncertainty in asset returns has two sources:
  - a common or macroeconomic factor
  - firm-specific events.
Factor Models of Security Returns

- The common factor is constructed to have zero expected value, since we use it to measure new information concerning the macroeconomy which, by definition, has zero expected value.
Suppose now that we call $F$ the deviation of the common factor from its expected value, $\beta_i$, the sensitivity of firm $i$ to that factor, and $e_i$, the firm-specific disturbance.
The factor model states that the actual return on firm $i$ will equal its initially expected return plus a (zero expected value) random amount attributable to unanticipated economywide events, plus another (zero expected value) random amount attributable to firm-specific events.
Formally, the single-factor model is described by equation 11.1:

\[ r_i = E(r_i) + \beta_i F + e_i \quad (11.1) \]

where \( E(r_i) \) is the expected return on stock \( i \).
Factor Models of Security Returns

- Notice that if the macro factor has a value of 0 in any particular period (i.e., no macro surprises), the return on the security will equal its previously expected value, \( E(r_i) \), plus the effect of firm-specific events only. All the nonsystematic components of returns, the \( e_i \)s, are uncorrelated among themselves and uncorrelated with the factor \( F \).
EXAMPLE 11.1: Factor Models

- To make the factor model more concrete, consider an example. Suppose that the macro factor, $F$, is taken to be news about the state of the business cycle, measured by the unexpected percentage change in gross domestic product (GDP), and that the consensus is that GDP will increase by 4% this year.
EXAMPLE 11.1: Factor Models

- Suppose also that a stock’s β value is 1.2. If GDP increases by only 3%, then the value of $F$ would be -1%, representing a 1% disappointment in actual growth versus expected growth.
EXAMPLE 11.1: Factor Models

- Given the stock’s beta value, this disappointment would translate into a return on the stock that is 1.2% lower than previously expected. This macro surprise, together with the firm-specific disturbance, $e_i$, determine the total departure of the stock’s return from its originally expected value.
Factor Models of Security Returns

- The factor model’s decomposition of return into systematic and firm-specific components is compelling, but confining systematic risk to a single factor is not.
Factor Models of Security Returns

- Indeed, when we motivated the index model in Chapter 10, we noted that the systematic or macro factor summarized by the market return arises from a number of sources, for example, uncertainty about the business cycle, interest rates, inflation, and so on. The market return reflects both macro factors as well as the average sensitivity of firms to those factors.
Factor Models of Security Returns

- When we estimate a single-index regression, therefore, we implicitly impose an (incorrect) assumption that each stock has the same relative sensitivity to each risk factor.
Factor Models of Security Returns

- If stocks actually differ in their betas relative to the various macroeconomic factors, then lumping all systematic sources of risk into one variable such as the return on the market index will ignore the nuances that better explain individual-stock returns.
Factor Models of Security Returns

- It stands to reason that a more explicit representation of systematic risk, allowing for the possibility that different stocks exhibit different sensitivities to its various components, would constitute a useful refinement of the single-factor model. It is easy to see that models that allow for several factors—multifactor models—can provide better descriptions of security returns.
Factor Models of Security Returns

- Apart from their use in building models of equilibrium security pricing, multifactor models are useful in risk management applications. These models give us a simple way to measure our exposure to various macroeconomic risks, and construct portfolios to hedge those risks.
Let’s start with a two-factor model. Suppose the two most important macroeconomic sources of risk are uncertainties surrounding the state of the business cycle, news of which we will again measure by unanticipated growth in GDP and changes in interest rates. We will call any unexpected decline in interest rates, which ought to be good news for stocks, IR.
The return on any stock will respond both to sources of macro risk as well as to its own firm-specific influences. We therefore can write a two-factor model describing the rate of return on stock \( i \) in some time period as follows:

\[
    r_i = E(r_i) + \beta_{iGDP}GDP + \beta_{iIR}IR + e_i \quad (11.2)
\]
The two macro factors on the right-hand side of the equation 11.2 comprise the systematic factors in the economy. As in the single-factor model, both of these macro factors have zero expectation: they represent changes in these variables that have not already been anticipated.
The coefficients of each factor in equation 11.2 measure the sensitivity of share returns to that factor. For this reason the coefficients are sometimes called factor sensitivities, factor loadings, or equivalently, factor betas. Also as before, $e_i$ reflects firm-specific influences.
Factor Models of Security Returns

- To illustrate the advantages of multifactor models in understanding the sources of macro risk, consider two firms, one a regulated electric-power utility in a mostly residential area, the other an airline.
Because residential demand for electricity is not very sensitive to the business cycle, the utility is likely to have a low beta on GDP. But the utility’s stock price may have a relatively high sensitivity to interest rates. Since the cash flow generated by the utility is relatively stable, its present value behaves much like that of a bond, varying inversely with interest rates.
Conversely, the performance of the airline is very sensitive to economic activity but is less sensitive to interest rates. It will have a high GDP beta and a lower interest rate beta.
Suppose that on a particular day, there is a piece of news suggesting that the economy will expand. GDP is expected to increase, but so are interest rates. Is the “macro news” on this day good or bad?
Factor Models of Security Returns

- For the utility, this is bad news: its dominant sensitivity is to rates. But for the airline, which responds more to GDP, this is good news. Clearly a one-factor or single-index model cannot capture such differential responses to varying sources of macroeconomic uncertainty.
EXAMPLE 11.2: Risk Assessment Using Multifactor Models

Suppose we estimate the two-factor model in equation 11.2 for Northeast Airlines and find the following result:

\[ r = 0.10 + 1.8(GDP) + 0.7(IR) + e \]
EXAMPLE 11.2: Risk Assessment Using Multifactor Models

- This tells us that based on currently available information, the expected rate of return for Northeast is 10%, but that for every percentage point increase in GDP beyond current expectations, the return on Northeast shares increases on average by 1.8%, while for every unanticipated percentage point that interest rates decreases, Northeast’s shares rise on average by .7%. 
EXAMPLE 11.2: Risk Assessment Using Multifactor Models

- The factor betas can provide a framework for a hedging strategy. The idea for an investor who wishes to hedge a source of risk is to establish an opposite factor exposure to offset that particular source of risk. Often, futures contracts can be used to hedge particular factor exposures. We explore this application in Chapter 22.
A Multifactor Security Market Line

- As it stands, the multifactor model is no more than a *description* of the factors that affect security returns. There is no “theory” in the equation.
The obvious question left unanswered by a factor model like equation 11.2 is where $E(r)$ comes from, in other words, what determines a security’s expected rate of return. This is where we need a theoretical model of equilibrium security returns.
A Multifactor Security Market Line

- In the previous two chapters we developed one example of such a model: the Security Market Line of the Capital Asset Pricing Model.
The CAPM asserts that securities will be priced to give investors an expected return comprised of two components: the risk-free rate, which is compensation for the time value of money, and a risk premium, determined by multiplying a benchmark risk premium (i.e., the risk premium offered by the market portfolio) times the relative measure of risk, beta.
The SML of the CAPM can be formally stated as follows:

\[ E(r) = r_f + \beta[E(r_M) - r_f] \]  \hspace{1cm} (11.3)
If we denote the risk premium of the market portfolio by $\text{RP}_M$, then a useful way to rewrite equation 11.3 is as follows:

$$E(r) = r_f + \beta \text{RP}_M \quad (11.4)$$
We pointed out in Chapter 10 that you can think of beta as measuring the exposure of a stock or portfolio to marketwide or macroeconomic risk factors.
Therefore, one interpretation of the SML is that investors are rewarded with a higher expected return for their exposure to macro risk, based on both the sensitivity to that risk (beta) as well as the compensation for bearing each unit of that source of risk (i.e., the risk premium, $R_{PM}$), but are not rewarded for exposure to firm-specific uncertainty (the residual term $e_i$ in equation 11.1).
A Multifactor Security Market Line

- Perhaps not surprisingly, a multifactor index model gives rise to a multifactor security market line in which the risk premium is determined by the exposure to each systematic risk factor, and by a risk premium associated with each of those factors.
A Multifactor Security Market Line

- For example, in a two-factor economy in which risk exposures can be measured by equation 11.2, we would conclude that the expected rate of return on a security would be the sum of: (next slide)
A Multifactor Security Market Line

- 1. The risk-free rate of return.
- 2. The sensitivity to GDP risk (i.e., the GDP beta) times the risk premium for GDP risk.
- 3. The sensitivity to interest rate risk (i.e., the interest rate beta) times the risk premium for interest rate risk.
This assertion is expressed as follows in equation 11.5. In that equation, for example, $\beta_{GDP}$ denotes the sensitivity of the security return to unexpected changes in GDP growth, and $RP_{GDP}$ is the risk premium associated with “one unit” of GDP exposure, i.e., the exposure corresponding to a GDP beta of 1.0.
A Multifactor Security Market Line

- Here then is a two-factor security market line:

\[ E(r) = r_f + \beta_{\text{GDP}} R_{\text{GDP}} + \beta_{\text{IR}} R_{\text{IR}} \]  \hspace{1cm} (11.5)
A Multifactor Security Market Line

- If you look back at equation 11.4, you will see that equation 11.5 is a generalization of the simple security market line. In the usual SML, the benchmark risk premium is given by the market portfolio, \( \text{RP}_M = E(r_M) - r_f \), but once we generalize to multiple risk sources, each with its own risk premium, we see that the insights are highly similar.
We still need to specify how to estimate the risk premium for each factor. Analogously to the simple CAPM, the risk premium associated with each factor can be thought of as the risk premium of a portfolio that has a beta of 1.0 on that particular factor and a beta of zero on all other factors.
In other words, it is the risk premium one might expect to earn by taking a “pure play” on that factor. We will return to this below, but for now, let’s just take the factor risk premia as given and see how a multifactor SML might be used.
EXAMPLE 11.3: A Multifactor SM

Think about our regression estimates for Northeast Airlines in Example 11.2. Northeast has a GDP beta of 1.8 and an interest rate beta of .7. Suppose the risk premium for one unit of exposure to GDP risk is 6%, while the risk premium for one unit of exposure to interest rate risk is 3%.
EXAMPLE 11.3: A Multifactor SM

- Then the overall risk premium on the Northeast portfolio should equal the sum of the risk premiums required as compensation for each source of systematic risk.
EXAMPLE 11.3: A Multifactor SM

- The risk premium attributable to GDP risk should be the stock’s exposure to that risk multiplied by the risk premium of the first factor portfolio, 6%. Therefore, the portion of the firm’s risk premium that is compensation for its exposure to the first factor is $1.8 \times 6\% = 10.8\%$. 
EXAMPLE 11.3: A Multifactor SM

- Similarly, the risk premium attributable to interest rate risk is \(0.7 \times 3\% = 2.1\%\). The total risk premium should be \(10.8\% + 2.1\% = 12.9\%\). Therefore, if the risk-free rate is 4%, the total return on the portfolio should be (next slide)
EXAMPLE 11.3: A Multifactor SM

- 4.0%  Risk-free rate
  + 10.8%  Risk premium for exposure to GDP risk
  + 2.1%  Risk premium for exposure to interest rate risk

16.9%  Total expected return
EXAMPLE 11.3: A Multifactor SM

- More concisely,

\[ E(r) = 4\% + 1.8\% \times 6\% + .7 \times 3\% = 16.9\% \]
The multifactor model clearly gives us a much richer way to think about risk exposures and compensation for those exposures than the single-index model or CAPM. Let us now fill in some of the gaps in the argument and more carefully explore the link between multifactor models of security returns and multifactor security market lines.
Stephen Ross developed the **arbitrage pricing theory** (APT) in 1976. Like the CAPM, the APT predicts a Security Market Line linking expected returns to risk, but the path it takes to its SML is quite different.
ARBITRAGE PRICING THEORY

- Ross’s APT relies on three key propositions:
  - (i) security returns can be described by a factor model;
  - (ii) there are sufficient securities to diversify away idiosyncratic risk; and
  - (iii) well-functioning security markets do not allow for the persistence of arbitrage opportunities.
We begin with a simple version of Ross’s model, which assumes that only one systematic factor affects security returns. However, the usual discussion of the APT is concerned with the multifactor case, so we treat this more general case as well.
Arbitrage, Risk Arbitrage, and Equilibrium

- An **arbitrage** opportunity arises when an investor can earn riskless profits without making a net investment.
- A trivial example of an arbitrage opportunity would arise if shares of a stock sold for different prices on two different exchanges.
Arbitrage, Risk Arbitrage, and Equilibrium

- For example, suppose IBM sold for $60 on the NYSE but only $58 on Nasdaq. Then you could buy the shares on Nasdaq and simultaneously sell them on the NYSE, clearing a riskless profit of $2 per share without tying up any of your own capital.
Arbitrage, Risk Arbitrage, and Equilibrium

- The Law of One Price states that if two assets are equivalent in all economically relevant respects, then they should have the same market price.
Arbitrage, Risk Arbitrage, and Equilibrium

- The Law of One Price is enforced by arbitrageurs: if they observe a violation of the law, they will engage in arbitrage activity—simultaneously buying the asset where it is cheap and selling where it is expensive. In the process, they will bid up the price where it is low and force it down where it is high until the arbitrage opportunity is eliminated.
Arbitrage, Risk Arbitrage, and Equilibrium

- The idea that market prices will move to rule out arbitrage opportunities is perhaps the most fundamental concept in capital market theory. Violation of this restriction would indicate the grossest form of market irrationality.
Arbitrage, Risk Arbitrage, and Equilibrium

- The critical property of a risk-free arbitrage portfolio is that any investor, regardless of risk aversion or wealth, will want to take an infinite position in it. Because those large positions will quickly force prices up or down until the opportunity vanishes, security prices should satisfy a “no-arbitrage condition,” that is, a condition that rules out the existence of arbitrage opportunities.
Arbitrage, Risk Arbitrage, and Equilibrium

- There is an important difference between arbitrage and risk-return dominance arguments in support of equilibrium price relationships. A dominance argument holds that when an equilibrium price relationship is violated, many investors will make limited portfolio changes, depending on their degree of risk aversion.
Arbitrage, Risk Arbitrage, and Equilibrium

- Aggregation of these limited portfolio changes is required to create a large volume of buying and selling, which in turn restores equilibrium prices.

- By contrast, when arbitrage opportunities exist each investor wants to take as large a position as possible; hence it will not take many investors to bring about the price pressures necessary to restore equilibrium.
Arbitrage, Risk Arbitrage, and Equilibrium

- Therefore, implications for prices derived from no-arbitrage arguments are stronger than implications derived from a risk-return dominance argument.
Arbitrage, Risk Arbitrage, and Equilibrium

- The CAPM is an example of a dominance argument, implying that all investors hold mean-variance efficient portfolios. If a security is mispriced, then investors will tilt their portfolios toward the underpriced and away from the overpriced securities.
Arbitrage, Risk Arbitrage, and Equilibrium

Pressure on equilibrium prices results from many investors shifting their portfolios, each by a relatively small dollar amount. The assumption that a large number of investors are mean-variance sensitive is critical.
Arbitrage, Risk Arbitrage, and Equilibrium

- In contrast, the implication of a no-arbitrage condition is that a few investors who identify an arbitrage opportunity will mobilize large dollar amounts and quickly restore equilibrium.
“Arbitrageur” often refers to a professional searching for mispriced securities in specific areas such as merger-target stocks, rather than to one who seeks strict (risk-free) arbitrage opportunities. Such activity is sometimes called risk arbitrage to distinguish it from pure arbitrage.
Well-Diversified Portfolios

- Now we look at the risk of a portfolio of stocks. We first show that if a portfolio is well diversified, its firm-specific or nonfactor risk becomes negligible, so that only factor (or systematic) risk remains.
Well-Diversified Portfolios

- If we construct an $n$-stock portfolio with weights $w_i$, $\sum w_i = 1$, then the rate of return on this portfolio is as follows:

$$r_p = E(r_p) + \beta_p F + e_p \quad (11.6)$$

where $\beta_p = \sum w_i \beta_i$ is the weighted average of the $\beta_i$ of the $n$ securities.
Well-Diversified Portfolios

- The portfolio nonsystematic component (which is uncorrelated with $F$) is $e_p = \sum w_i e_i$, which similarly is a weighted average of the $e_i$ of the n securities.
Well-Diversified Portfolios

- We can divide the variance of this portfolio into systematic and nonsystematic sources, as we saw in Chapter 10.
Well-Diversified Portfolios

- The portfolio variance is

\[ \sigma_P^2 = \beta_P^2 \sigma_F^2 + \sigma^2(e_P) \]

where \( \sigma_F^2 \) is the variance of the factor \( F \) and \( \sigma^2(e_P) \) is the nonsystematic risk of the portfolio, which is given by

\[ \sigma^2(e_P) = \text{Variance}(\sum w_i e_i) = \sum w_i^2 \sigma^2(e_i) \]
Well-Diversified Portfolios

- Note that in deriving the nonsystematic variance of the portfolio, we depend on the facts that the firm-specific $e_i$s are uncorrelated and hence that the variance of the “portfolio” of nonsystematic $e_i$s is the weighted sum of the individual nonsystematic variances with the square of the investment proportions as weights.
Well-Diversified Portfolios

If the portfolio were equally weighted, \( w_i = 1/n \), then the nonsystematic variance would be

\[
\sigma^2(e_P) = \text{Variance}(\sum w_i e_i)
\]

\[
= \sum \left( \frac{1}{n} \right)^2 \sigma^2(e_i) = \frac{1}{n} \sum \frac{\sigma^2(e_i)}{n} = \frac{1}{n} \bar{\sigma}^2(e_i)
\]

where the last term is the average value across securities of nonsystematic variance.
Well-Diversified Portfolios

- In words, the nonsystematic variance of the portfolio equals the average nonsystematic variance divided by $n$. Therefore, when the portfolio gets large in the sense that $n$ is large, its nonsystematic variance approaches zero. This is the effect of diversification.
We conclude that for the equally weighted portfolio, the nonsystematic variance approaches zero as \( n \) becomes ever larger. This property is true of portfolios other than the equally weighted one.
Well-Diversified Portfolios

- Any portfolio for which each $w_i$ becomes consistently smaller as $n$ gets large (more precisely, for which each $w_i^2$ approaches zero as $n$ increases) will satisfy the condition that the portfolio nonsystematic risk will approach zero.
In fact, this property motivates us to define a well-diversified portfolio as one that is diversified over a large enough number of securities with each weight, $w_i$, small enough that for practical purposes the nonsystematic variance, $\sigma^2(e_P)$, is negligible.
Well-Diversified Portfolios

- Because the expected value of $e_P$ for any well-diversified portfolio is zero, and its variance also is effectively zero, we can conclude that any realized value of $e_P$ will be virtually zero.
Well-Diversified Portfolios

- Rewriting equation 11.1, we conclude that for a well-diversified portfolio, for all practical purposes

\[ r_P = E(r_P) + \beta_P F \]
Well-Diversified Portfolios

- Large (mostly institutional) investors can hold portfolios of hundreds and even thousands of securities; thus the concept of well-diversified portfolios clearly is operational in contemporary financial markets.
Betas and Expected Returns

- Because nonfactor risk can be diversified away, only factor risk should command a risk premium in market equilibrium.
Betas and Expected Returns

- Nonsystematic risk across firms cancels out in well-diversified portfolios; one would not expect investors to be rewarded for bearing risk that can be eliminated through diversification. Instead, only the systematic risk of a portfolio of securities should be related to its expected returns.
Betas and Expected Returns

- The solid line in Figure 11.1A plots the return of a well-diversified Portfolio A with $\beta_A = 1$ for various realizations of the systematic factor.
Figure 11.1
Betas and Expected Returns

- The expected return of Portfolio A is 10%; this is where the solid line crosses the vertical axis. At this point the systematic factor is zero, implying no macro surprises.

- If the macro factor is positive, the portfolio’s return exceeds its expected value; if it is negative, the portfolio’s return falls short of its mean.
Betas and Expected Returns

- The return on the portfolio is therefore

\[ E(r_A) + \beta_A F = 10\% + 1.0 \times F \]

Compare Figure 11.1A with Figure 11.1B, which is a similar graph for a single stock \((S)\) with \(\beta_s = 1\).
Betas and Expected Returns

- The undiversified stock is subject to nonsystematic risk, which is seen in a scatter of points around the line. The well-diversified portfolio’s return, in contrast, is determined completely by the systematic factor.
Betas and Expected Returns

- Now consider Figure 11.2, where the dashed line plots the return on another well-diversified portfolio, Portfolio $B$, with an expected return of 8% and $\beta_B$ also equal to 1.0.
Figure 11.2
Returns as a function of the systematic factor: an arbitrage opportunity
Betas and Expected Returns

- Could Portfolios A and B coexist with the return pattern depicted? Clearly not: No matter what the systematic factor turns out to be. Portfolio A outperforms Portfolio B, leading to an arbitrage opportunity.
Betas and Expected Returns

- If you sell short $1 million of $B$ and buy $1$ million of $A$, a zero net investment strategy, your riskless payoff would be $20,000$, as follows:

Long position in $A$: $(.10 + 1.0 \times F) \times $1 million

Short position in $B$: $-(.08 + 1.0 \times F) \times $1 million

$\Rightarrow$ Net proceeds: $.02 \times $1 million = $20,000
Your profit is risk-free because the factor risk cancels out across the long and short positions. Moreover, the strategy requires zero net investment. You should pursue it on an infinitely large scale until the return discrepancy between the two portfolios disappears.
Betas and Expected Returns

- Well-diversified portfolios with equal betas must have equal expected returns in market equilibrium, or arbitrage opportunities exist.
Betas and Expected Returns

- What about portfolios with different betas? We show now that their risk premiums must be proportional to beta. To see why, consider Figure 11.3.
Figure 11.3
An arbitrage opportunity

Expected Return (%)

β (With respect to macro factor)

Risk Premium

$\mathbf{r}_f = 4$

D

C

A

F

100
Suppose that the risk-free rate is 4% and that a well-diversified portfolio, C, with a beta of .5, has an expected return of 6%. Portfolio C plots below the line from the risk-free asset to Portfolio A.
Betas and Expected Returns

- Consider, therefore, a new portfolio, $D$, composed of half of Portfolio $A$ and half of the risk-free asset. Portfolio $D$’s beta will be $(.5 \times 0 + .5 \times 1.0) = .5$, and its expected return will be $(.5 \times 4 + .5 \times 10) = 7\%$. 
Betas and Expected Returns

- Now Portfolio $D$ has an equal beta but a greater expected return than Portfolio $C$. From our analysis in the previous paragraph we know that this constitutes an arbitrage opportunity.
Betas and Expected Returns

- We conclude that, to preclude arbitrage opportunities, the expected return on all well-diversified portfolios must lie on the straight line from the risk-free asset in Figure 11.3. The equation of this line will dictate the expected return on all well-diversified portfolios.
Betas and Expected Returns

Notice in Figure 11.3 that risk premiums are indeed proportional to portfolio betas. The risk premium is depicted by the vertical arrow, which measures the distance between the risk-free rate and the expected return on the portfolio. The risk premium is zero for $\beta = 0$ and rises in direct proportion to $\beta$. 
The One-Factor Security Market Line

- Now consider the market index portfolio, $M$, as a well-diversified portfolio, and let us measure the systematic factor as the unexpected return on that portfolio. Because the index portfolio must be on the line in Figure 11.4 and the beta of the index portfolio is 1, we can determine the equation describing that line.
Figure 11.4
The security market line
As Figure 11.4 shows, the intercept is $r_f$ and the slope is $E(r_M) - r_f$ \[\text{rise} = E(r_M) - r_f; \text{run} = 1\], implying that the equation of the line is

$$E(r_P) = r_f + [E(r_M) - r_f] \beta_P \quad (11.7)$$

Hence, Figures 11.3 and 11.4 imply an SML relation equivalent to that of the CAPM.
EXAMPLE 11.4: Arbitrage and the Security Market Line

Suppose the market index is a well-diversified portfolio with expected return 10% and that deviations of its return from expectation (i.e., $r_M - 10\%$) can serve as the systematic factor. The T-bill rate is 4%.
EXAMPLE 11.4: Arbitrage and the Security Market Line

- Then the SML (equation 11.7) implies that the expected rate of return on well-diversified Portfolio E with a beta of 2/3 should be $4\% + \left(\frac{2}{3}\right)(10 - 4) = 8\%$. What if its expected return actually is 9%? Then there will be an arbitrage opportunity.
EXAMPLE 11.4: Arbitrage and the Security Market Line

- Buy $1 of the stock and sell $1 of a portfolio that is invested 1/3 in T-bills and 2/3 in the market. This portfolio by construction has the same beta as Portfolio $E$. The return on this portfolio is $(1/3) \times r_f + (2/3) \times r_M = (1/3) \times 4\% + (2/3) \times r_M$. 
EXAMPLE 11.4: Arbitrage and the Security Market Line

-The net return on the combined position is:

Invest $1 in portfolio E, with expected return 9% and beta of (2/3) on surprise in market return: $1 \times [0.09 + \frac{2}{3}(r_M - 0.10)]

Sell portfolio invested (1/3) in T-bills and (2/3) in the market index: $-1 \times \left[\frac{1}{3} \times 4\% + \frac{2}{3} \times r_M\right]

\Rightarrow \text{Total: } $1 \times 0.01
EXAMPLE 11.4: Arbitrage and the Security Market Line

- The profit per dollar invested is risk-free and precisely equal to the deviation of expected return from the SML.
The One-Factor Security Market Line

- We have used the no-arbitrage condition to obtain an expected return–beta relationship identical to that of the CAPM, without the restrictive assumptions of the CAPM.
As noted, this derivation depends on three assumptions: a factor model describing security returns, a sufficient number of securities to form well-diversified portfolios, and the absence of arbitrage opportunities. This last restriction gives rise to the name of the approach: Arbitrage Pricing Theory.
Our demonstration suggests that despite its restrictive assumptions, the main conclusion of the CAPM, namely, the SML expected return-beta relationship, should be at least approximately valid.
It is worth noting that in contrast to the CAPM, the APT does not require that the benchmark portfolio in the SML relationship be the true market portfolio. Any well-diversified portfolio lying on the SML of Figure 11.4 may serve as the benchmark portfolio.
The One-Factor Security Market Line

- For example, one might define the benchmark portfolio as the well-diversified portfolio most highly correlated with whatever systematic factor is thought to affect stock returns. Accordingly, the APT has more flexibility than does the CAPM because problems associated with an unobservable market portfolio are not a concern.
The One-Factor Security Market Line

- In addition, the APT provides further justification for use of the index model in the practical implementation of the SML relationship.
Even if the index portfolio is not a precise proxy for the true market portfolio, which is a cause of considerable concern in the context of the CAPM, we now know that if the index portfolio is sufficiently well diversified, the SML relationship should still hold true according to the APT.
So far we have demonstrated the APT relationship for well-diversified portfolios only. The CAPM expected return–beta relationship applies to single assets, as well as to portfolios. In the next section we generalize the APT result one step further.
INDIVIDUAL ASSETS AND THE APT

- We have demonstrated that if arbitrage opportunities are to be ruled out, each well-diversified portfolio’s expected excess return must be proportional to its beta.
- The question is whether this relationship tells us anything about the expected returns on the component stocks.
The answer is that if this relationship is to be satisfied by all well-diversified portfolios, it must be satisfied by *almost* all individual securities, although a full proof of this proposition is somewhat difficult. We can illustrate the argument less formally.
Suppose that the expected return–beta relationship is violated for all single assets. Now create a pair of well-diversified portfolios from these assets. What are the chances that in spite of the fact that for any pair of assets the relationship does not hold, the relationship will hold for both well-diversified portfolios?
INDIVIDUAL ASSETS AND THE APT

- The chances are small, but it is possible that the relationships among the single securities are violated in offsetting ways so that somehow it holds for the pair of well-diversified portfolios.
INDIVIDUAL ASSETS AND THE APT

- Now construct yet a third well-diversified portfolio. What are the chances that the violations of the relationships for single securities are such that the third portfolio also will fulfill the no-arbitrage expected return–beta relationship?
INDIVIDUAL ASSETS AND THE APT

- Obviously, the chances are smaller still, but the relationship is possible. Continue with a fourth well-diversified portfolio, and so on.
INDIVIDUAL ASSETS AND THE APT

- If the no-arbitrage expected return–beta relationship has to hold for infinitely many different, well-diversified portfolios, it must be virtually certain that the relationship holds for all but a small number of individual securities.
INDIVIDUAL ASSETS AND THE APT

- We use the term *virtually certain* advisedly because we must distinguish this conclusion from the statement that all securities surely fulfill this relationship. The reason we cannot make the latter statement has to do with a property of well-diversified portfolios.
Recall that to qualify as well diversified, a portfolio must have very small positions in all securities. If, for example, only one security violates the expected return–beta relationship, then the effect of this violation on a well-diversified portfolio will be too small to be of importance for any practical purpose, and meaningful arbitrage opportunities will not arise.
INDIVIDUAL ASSETS AND THE APT

- But if many securities violate the expected return-beta relationship, the relationship will no longer hold for well-diversified portfolios, and arbitrage opportunities will be available.
Consequently, we conclude that imposing the no-arbitrage condition on a single-factor security market implies maintenance of the expected return–beta relationship for all well-diversified portfolios and for all but possibly a small number of individual securities.
The APT and the CAPM

- The APT serves many of the same functions as the CAPM. It gives us a benchmark for rates of return that can be used in capital budgeting, security evaluation, or investment performance evaluation.
Moreover, the APT highlights the crucial distinction between nondiversifiable risk (factor risk) that requires a reward in the form of a risk premium and diversifiable risk that does not.
The APT and the CAPM

- The APT depends on the assumption that a rational equilibrium in capital markets precludes arbitrage opportunities.
- A violation of the APT’s pricing relationships will cause extremely strong pressure to restore them even if only a limited number of investors become aware of the disequilibrium.
The APT and the CAPM

Furthermore, the APT yields an expected return–beta relationship using a well-diversified portfolio that practically can be constructed from a large number of securities.
In contrast, the CAPM is derived assuming an inherently unobservable “market” portfolio. The CAPM argument rests on mean-variance efficiency; that is, if any security violates the expected return–beta relationship, then many investors (each relatively small) will tilt their portfolios so that their combined overall pressure on prices will restore an equilibrium that satisfies the relationship.
The APT and the CAPM

- In spite of these apparent advantages, the APT does not fully dominate the CAPM. The CAPM provides an unequivocal statement on the expected return–beta relationship for all securities, whereas the APT implies that this relationship holds for all but perhaps a small number of securities.
The APT and the CAPM

Because it focuses on the no-arbitrage condition, without the further assumptions of the market or index model, the APT cannot rule out a violation of the expected return–beta relationship for any particular asset. For this, we need the CAPM assumptions and its dominance arguments.
A MULTIFACTOR APT

- We can derive a multifactor version of the APT to accommodate the multiple sources of risk. Suppose that we generalize the factor model expressed in equation 11.1 to a two-factor model:

\[ r_i = E(r_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + e_i \]  

(11.8)
In Example 11.2, Factor 1 was the departure of GDP growth from expectations, and Factor 2 was the unanticipated decline in interest rates. Each factor has zero expected value because each measures the surprise in the systematic variable rather than the level of the variable.
Similarly, the firm-specific component of unexpected return, $e_i$, also has zero expected value. Extending such a two-factor model to any number of factors is straightforward.
A MULTIFACTOR APT

- Establishing a multifactor APT is similar to the one-factor case. But first we must introduce the concept of a factor portfolio, which is a well-diversified portfolio constructed to have a beta of 1 on one of the factors and a beta of 0 on any other factor. We can think of a factor portfolio as a tracking portfolio.
A MULTIFACTOR APT

- That is, the returns on such a portfolio track the evolution of particular sources of macroeconomic risk, but are uncorrelated with other sources of risk.
A MULTIFACTOR APT

- It is possible to form such factor portfolios because we have a large number of securities to choose from, and a relatively small number of factors. Factor portfolios will serve as the benchmark portfolios for a multifactor security market line.
EXAMPLE 11.5: Multifactor SML

- Suppose that the two factor portfolios, Portfolios 1 and 2, have expected returns $E(r_1) = 10\%$ and $E(r_2) = 12\%$. Suppose further that the risk-free rate is 4%. The risk premium on the first factor portfolio is $10\% - 4\% = 6\%$, whereas that on the second factor portfolio is $12\% - 4\% = 8\%$. 
EXAMPLE 11.5: Multifactor SML

Now consider a well-diversified portfolio, Portfolio A, with beta on the first factor, $\beta_{A1} = .5$, and beta on the second factor, $\beta_{A2} = .75$. The multifactor APT states that the overall risk premium on this portfolio must equal the sum of the risk premiums required as compensation for each source of systematic risk.
EXAMPLE 11.5: Multifactor SML

- The risk premium attributable to risk factor 1 should be the portfolio’s exposure to factor 1, $\beta_{A1}$, multiplied by the risk premium earned on the first factor portfolio, $E(r_i) - r_f$. 
EXAMPLE 11.5: Multifactor SML

Therefore, the portion of Portfolio A’s risk premium that is compensation for its exposure to the first factor is \( \beta_{A1}[E(r_1) - r_f] = 0.5(10\% - 4\%) = 3\% \), whereas the risk premium attributable to risk factor 2 is \( \beta_{A2}[E(r_2) - r_f] = 0.75(12\% - 4\%) = 6\% \). The total risk premium on the portfolio should be 3\% + 6\% = 9\% and the total return on the portfolio should be 4\% + 9\% = 13\%. 
EXAMPLE 11.5: Multifactor SML

To generalize the argument in Example 11.5, note that the factor exposures of any portfolio, $P$, are given by its betas, $\beta_{P1}$ and $\beta_{P2}$. A competing portfolio, $Q$, can be formed by investing in factor portfolios with the following weights: $\beta_{P1}$ in the first factor portfolio, $\beta_{P2}$ in the second factor portfolio, and $1 - \beta_{P1} - \beta_{P2}$ in T-bills.
EXAMPLE 11.5: Multifactor SML

- By construction, portfolio \( Q \) will have betas equal to those of Portfolio \( P \) and expected return of

\[
E(r_Q) = \beta_{P_1}E(r_1) + \beta_{P_2}E(r_2) + (1 - \beta_{P_1} - \beta_{P_2})r_f
= r_f + \beta_{P_1}[E(r_1) - r_f] + \beta_{P_2}[E(r_2) - r_f]
\]

(11.9)
EXAMPLE 11.5: Multifactor SML

- Using the numbers in Example 11.5:

\[ E(r_Q) = 4\% + 0.5(10\% - 4\%) + 0.75 \times (12\% - 4\%) = 13\% \]
Because Portfolio Q has precisely the same exposures as Portfolio A to the two sources of risk, their expected returns also ought to be equal. So Portfolio A also ought to have an expected return of 13%. If it does not, then there will be an arbitrage opportunity.
EXAMPLE 11.6: Mispricing and Arbitrage

Suppose that the expected return on Portfolio A were 12% rather than 13%. This return would give rise to an arbitrage opportunity. Form a portfolio from the factor portfolios with the same betas as Portfolio A. This requires weights of .5 on the first factor portfolio, .75 on the second factor portfolio, and -.25 on the risk-free asset.
EXAMPLE 11.6: Mispricing and Arbitrage

- This portfolio has exactly the same factor betas as Portfolio A: It has a beta of .5 on the first factor because of its .5 weight on the first factor portfolio, and a beta of .75 on the second factor. (The weight of -.25 on risk-free T-bills does not affect the sensitivity to either factor.)
EXAMPLE 11.6: Mispricing and Arbitrage

- Now invest $1 in Portfolio $Q$ and sell (short) $1 in Portfolio $A$. Your net investment is zero, but your expected dollar profit is positive and equal to

\[ \$1 \times E(r_Q) - \$1 \times E(r_A) \]
\[ \begin{align*}
&= \$1 \times .13 - \$1 \times .12 \\
&= .01
\end{align*} \]
EXAMPLE 11.6: Mispricing and Arbitrage

Moreover, your net position is riskless. Your exposure to each risk factor cancels out because you are long $1 in Portfolio Q and short $1 in Portfolio A, and both of these well-diversified portfolios have exactly the same factor betas.
EXAMPLE 11.6: Mispricing and Arbitrage

- Thus, if Portfolio A’s expected return differs from that of Portfolio Q’s, you can earn positive risk-free profits on a zero net investment position. This is an arbitrage opportunity.
The APT and the CAPM

- We conclude that any well-diversified portfolio with betas $\beta_{p1}$ and $\beta_{p2}$ must have the return given in equation 11.9 if arbitrage opportunities are to be precluded. If you compare equations 11.3 and 11.9, you will see that equation 11.9 is simply a generalization of the one-factor SML.
The APT and the CAPM

- Finally, the extension of the multifactor SML of equation 11.9 to individual assets is precisely the same as for the one-factor APT. Equation 11.9 cannot be satisfied by every well-diversified portfolio unless it is satisfied by virtually every security taken individually. Equation 11.9 thus presents the multifactor SML for an economy with multiple sources of risk.
The APT and the CAPM

- We pointed out earlier that one application of the CAPM is to provide “fair” rates of return for regulated utilities. The multifactor APT can be used to the same ends. The nearby box summarizes a study in which the APT was applied to find the cost of capital for regulated electric companies.
WHERE SHOULD WE LOOK FOR FACTORS?

- One shortcoming of the multifactor APT is that it gives no guidance concerning the determination of the relevant risk factors or their risk premiums.
WHERE SHOULD WE LOOK FOR FACTORS?

Two principles guide us when we specify a reasonable list of factors.

- First, we want to limit ourselves to systematic factors with considerable ability to explain security returns. If our model calls for hundreds of explanatory variables, it does little to simplify our description of security returns.
WHERE SHOULD WE LOOK FOR FACTORS?

- Second, we wish to choose factors that seem likely to be important risk factors, i.e., factors that concern investors sufficiently that they will demand meaningful risk premiums to bear exposure to those sources of risk.
WHERE SHOULD WE LOOK FOR FACTORS?

- One example of the multifactor approach is the work of Chen, Roll, and Ross (JB, 1986) who chose the following set of factors based on the ability of these factors to paint a broad picture of the macroeconomy. Their set is obviously but one of many possible sets that might be considered.
WHERE SHOULD WE LOOK FOR FACTORS?

- IP = %change in industrial production
- EI = %change in expected inflation
- UI = %change in unanticipated inflation
- CG = excess return of long-term corporate bonds over long-term government bonds
- GB = excess return of long-term government bonds over T-bills
WHERE SHOULD WE LOOK FOR FACTORS?

- This list gives rise to the following five-factor model of security returns during holding period $t$ as a function of the change in the set of macroeconomic indicators:

$$ r_{it} = \alpha_i + \beta_{iIP}IP_t + \beta_{iEI}EI_t + \beta_{iCG}CG_t + \beta_{iGB}GB_t + e_{it} \quad (11.10) $$
WHERE SHOULD WE LOOK FOR FACTORS?

- Equation 11.10 is a multidimensional security characteristic line (SCL), with five factors. As before, to estimate the betas of a given stock we can use regression analysis. Here, however, because there is more than one factor, we estimate a *multiple* regression of the returns of the stock in each period on the five macroeconomic factors.
WHERE SHOULD WE LOOK FOR FACTORS?

- The residual variance of the regression estimates the firm-specific risk. We discuss the results of this model in the next chapter, which focuses on empirical evidence on security pricing.
WHERE SHOULD WE LOOK FOR FACTORS?

- An alternative approach to specifying macroeconomic factors as candidates for relevant sources of systematic risk uses firm characteristics that seem on empirical grounds to proxy for exposure to systematic risk. In other words, the factors are chosen as variables that on past evidence seem to predict high average returns and therefore may be capturing risk premiums.
WHERE SHOULD WE LOOK FOR FACTORS?

- One example of this approach is the so-called Fama and French three-factor model,

\[ r_{it} = \alpha_i + \beta_{iM} R_{Mt} + \beta_{iSMB} SMB_t + \beta_{iHML} HML_t + e_{it} \quad (11.11) \]
WHERE SHOULD WE LOOK FOR FACTORS?

- In equation (11.11),
  - SMB = Small Minus Big, i.e., the return of a portfolio of small stocks in excess of the return on a portfolio of large stocks
  - HML = High Minus Low, i.e., the return of a portfolio of stocks with a high book-to-market ratio in excess of the return on a portfolio of stocks with a low book-to-market ratio
WHERE SHOULD WE LOOK FOR FACTORS?

- Note that in the model of equation (11.11) the market index does play a role and is expected to capture systematic risk originating from macroeconomic factors.
WHERE SHOULD WE LOOK FOR FACTORS?

- These two firm-characteristic variables are chosen because of long-standing observations that corporate capitalization (firm size) and book-to-market ratio seem to be predictive of average stock returns.
WHERE SHOULD WE LOOK FOR FACTORS?

- Fama and French justify this model on empirical grounds: while SMB and HML are not themselves obvious candidates for relevant risk factors, the hope is that these variables proxy for yet-unknown more-fundamental variables.
WHERE SHOULD WE LOOK FOR FACTORS?

- For example, Fama and French point out that firms with high ratios of book to market value are more likely to be in financial distress and that small stocks may be more sensitive to changes in business conditions. Thus, these variables may capture sensitivity to risk factors in the macroeconomy.
WHERE SHOULD WE LOOK FOR FACTORS?

- The problem with empirical approaches such as the Fama-French model, which use proxies for extramarket sources of risk, is that none of the factors in the proposed models can be clearly identified as hedging a significant source of uncertainty.
Black points out that when researchers scan and rescan the database of security returns in search of explanatory factors (an activity often called data-snooping), they may eventually uncover past “patterns” that are due purely to chance. Black observes that return premiums to factors such as firm size have largely vanished since first discovered.
WHERE SHOULD WE LOOK FOR FACTORS?

- However, Fama and French point out that size and book-to-market ratios have predicted average returns in various time periods and in markets all over the world, thus mitigating potential effects of data-snooping.
A MULTIFACTOR CAPM

- The CAPM presupposes that the only relevant source of risk arises from variations in security returns, and therefore a representative (market) portfolio can capture this entire risk.
A MULTIFACTOR CAPM

- As a result, individual-stock risk can be defined by the contribution to overall portfolio risk; hence, the risk premium on an individual stock is solely determined by its beta on the market portfolio. But is this narrow view of risk warranted?
Consider a relatively young investor whose future wealth is determined in large part by labor income. The stream of future labor income is also risk and may be intimately tied to the fortunes of the company for which the investor works. Such an investor might choose an investment portfolio that will help to diversify labor-income risk.
A MULTIFACTOR CAPM

- For that purpose, stocks with lower-than-average correlation with future labor income would be favored, that is, such stocks will receive higher weights in the individual portfolio than their weights in the market portfolio.
A MULTIFACTOR CAPM

- Put another way, using this broader notion of risk, these investors no longer consider the market portfolio as efficient and the rationale for the CAPM expected return-beta relationship no longer applies.
A MULTIFACTOR CAPM

- In principle, the CAPM may still hold if the hedging demands of various investors are equally distributed across different types of securities so that deviations of portfolio weights from those of the market portfolio are offsetting.
A MULTIFACTOR CAPM

- But if hedging demands are common to many investors, the prices of securities with desirable hedging characteristics will be bid up and the expected return reduced, which will invalidate the CAPM expected return–beta relationship.
A MULTIFACTOR CAPM

- For example, suppose the prices of energy stocks were driven up by investors who buy such stocks to hedge uncertainty about energy expenditures. At those higher stock prices, expected rates of return will be lower than dictated by the expected return–beta relationship of the CAPM.
A MULTIFACTOR CAPM

- The simple SML relationship needs to be generalized to account for the effects of extramarket hedging demands on equilibrium rates of return.
Merton has shown that these hedging demands will result in an expanded or multifactor version of the CAPM that recognizes the multidimensional nature of risk. His model is called the multifactor CAPM or, alternatively, the Intertemporal CAPM (ICAPM for short).
A MULTIFACTOR CAPM

- The focal point of Merton’s model is not dollar returns per se, but the consumption and investment made possible by the investor’s wealth. Each source of risk to consumption or investment opportunities may in principle command its own risk premium.
In the case of energy price risk, for example, Merton’s model would imply that the expected return-beta relationship of the single-factor CAPM would be generalized to the following two-factor relationship:

\[ E(r_i) = r_f + \beta_{iM}[E(r_M) - r_f] + \beta_{ie}[E(r_e) - r_f] \]
A MULTIFACTOR CAPM

- In the above equation, $\beta_{iM}$ is the beta of security $i$ with respect to the market portfolio, and $\beta_{ie}$ is the beta with respect to energy price risk.
Similarly, $E(r_e) - r_f$ is the risk premium associated with exposure to energy price uncertainty. The rate of return of the portfolio that best hedges energy price uncertainty is $r_e$. This equation, therefore, is a two-factor CAPM. More generally, we will have a beta and a risk premium for every significant source of risk that consumers try to hedge.
A MULTIFACTOR CAPM

- Notice that this expanded version of the CAPM provides a prediction for security returns identical to that of the multifactor APT. Therefore, there is no contradiction between these two theories of the risk premium.
A MULTIFACTOR CAPM

- The CAPM approach does offer one notable advantage, however. In contrast to the APT, which is silent on the relevant systematic factors, the CAPM provides guidance as to where to look for those factors.
A MULTIFACTOR CAPM

- The important factors will be those sources of risk that large groups of investors try to offset by establishing extramarket hedge portfolios. By specifying the likely sources of risk against which dominant groups of investors attempt to hedge, we identify the dimensions along which the CAPM needs to be generalized.
A MULTIFACTOR CAPM

- When a source of risk has an effect on expected returns, we say that this risk “is priced.” While the single-factor CAPM predicts that only market risk will be priced, the ICAPM predicts that other sources of risk also may be priced.
Merton suggested a list of possible common sources of uncertainty that might affect expected security returns. Among these are uncertainties in labor income, prices of important consumption goods (e.g., energy prices), or changes in future investment opportunities (e.g., changes in the riskiness of various asset classes).
A MULTIFACTOR CAPM

- However, it is difficult to predict whether there exists sufficient demand for hedging these sources of uncertainty to affect security returns.